Entropy stable schemes for compressible flows on unstructured meshes

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The 2-D Navier-Stokes equations

$$\frac{\mathrm{d}\mathbf{U}}{\mathrm{d}t} + \frac{\partial \mathbf{f}_1}{\partial x} + \frac{\partial \mathbf{f}_2}{\partial y} = \frac{\partial \mathbf{g}_1}{\partial x} + \frac{\partial \mathbf{g}_2}{\partial y} \tag{1}$$

$$\mathbf{U} = \begin{bmatrix} \rho \\ \rho u_1 \\ \rho u_2 \\ E \end{bmatrix}, \quad \mathbf{f}_\alpha(\mathbf{U}) = \begin{bmatrix} \rho u_\alpha \\ \rho u_1 u_\alpha + p\delta_{1\alpha} \\ \rho u_2 u_\alpha + p\delta_{2\alpha} \\ (E+p)u_\alpha \end{bmatrix}, \quad \mathbf{g}_\alpha(\mathbf{U}) = \begin{bmatrix} 0 \\ \tau_{\alpha 1} \\ \tau_{\alpha 2} \\ u_1\tau_{\alpha 1} + u_2\tau_{\alpha 2} - Q_\alpha \end{bmatrix}$$

Here,

$\rho \to$ density $p \to$ pressure $\mathbf{u} = (u_1, u_2)^\top \to$ velocity

$E \to$ total energy $\boldsymbol{\tau} = \left[\tau_{\alpha\beta}\right] \to$ shear stress $\mathbf{Q} = (Q_1, Q_2) \to$ heat flux

Ignoring the viscous fluxes on the right gives us the **Euler equations**.

Entropy conditions

Entropy conditions are essential to single out a physically relevant solution. A conservation law equipped with a convex entropy function $\eta(\mathbf{U})$ and entropy fluxes $q_1(\mathbf{U}), q_2(\mathbf{U})$, satisfies the entropy inequality

$$\frac{\mathrm{d}\eta}{\mathrm{d}t} + \frac{\partial q_1(\mathbf{U})}{\partial x} + \frac{\partial q_2(\mathbf{U})}{\partial y} \le 0 \tag{2}$$

with the **entropy variables** $\mathbf{V} = \eta'(\mathbf{U})$. For the Euler equations, we choose

$$\eta(\mathbf{U}) = -\frac{\rho s}{\gamma - 1}, \qquad q_\alpha(\mathbf{U}) = -\frac{\rho u_\alpha s}{\gamma - 1}, \qquad s = \ln\left(\frac{p}{\rho^\gamma}\right) \tag{3}$$

The viscous fluxes of the Navier-Stokes system are written in terms of $\mathbf{V}$

$$\mathbf{g}_\alpha = \mathbf{K}_{\alpha 1}(\mathbf{V})\frac{\partial \mathbf{V}}{\partial x} + \mathbf{K}_{\alpha 2}(\mathbf{V})\frac{\partial \mathbf{V}}{\partial y}, \qquad \mathbf{K} = \left[\mathbf{K}_{\alpha\beta}\right] \ge 0 \tag{4}$$

This leads to a global entropy estimate

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \eta \,\mathrm{d}\Omega \le -\int_{\partial\Omega} (q_1 n_1 + q_2 n_2)\,\mathrm{d}S + \int_{\partial\Omega} \mathbf{V}^\top(\mathbf{g}_1 n_1 + \mathbf{g}_2 n_2)\,\mathrm{d}S \tag{5}$$

Objectives

- Design high-order semi-discrete finite volume schemes for the Euler equations satisfying a discrete version of (2) on unstructured meshes.
- Discretise the viscous fluxes appropriately to obtain a discrete global estimate analogous to (5).

Past work

Tadmor [1] proposed the idea of constructing **entropy conservative** schemes for conservation laws, to which numerical dissipation is added to obtain entropy stability. Higher order dissipation operators [3] have been obtained on structured meshes, by suitable reconstructions satisfying the **sign property**. First order entropy stable schemes have been designed on unstructured meshes [2].

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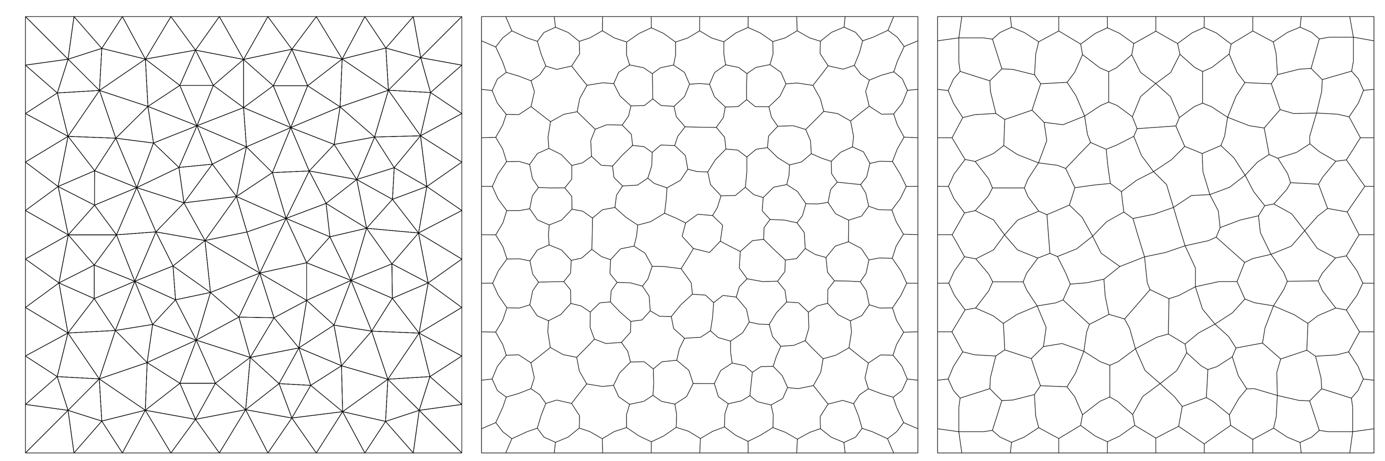
Methodology

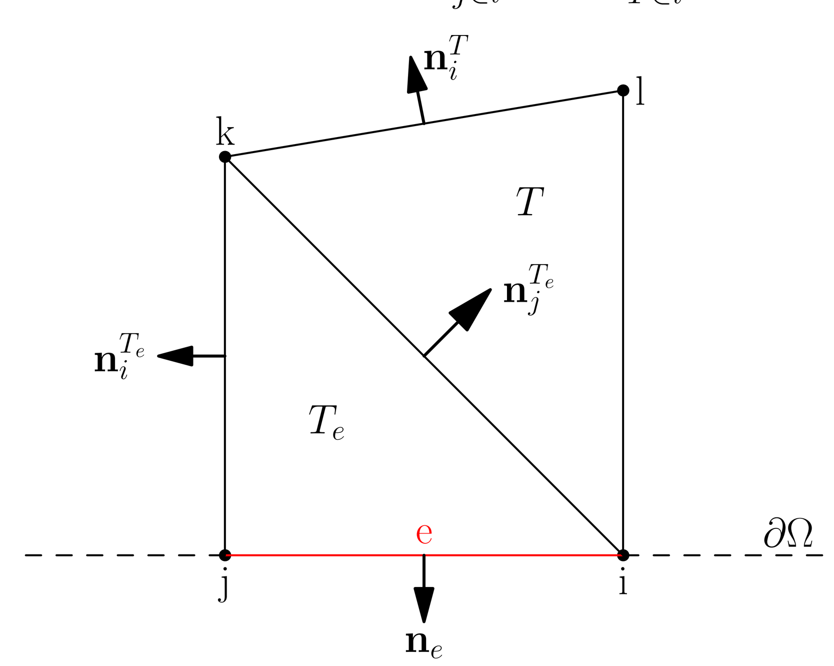


(a) (b) (c)

Figure 1: Unstructured meshes (a) Primary (b) Median dual (c) Voronoi dual

Consider the vertex-centered, semi-discrete finite volume scheme on dual meshes

$$|C_i|\frac{\mathrm{d}\mathbf{U}_i}{\mathrm{d}t} = -\sum_{j\in i}\mathbf{F}_{ij} + \sum_{T\in i}\mathbf{G}^T.\frac{\mathbf{n}_i^T}{2} - \sum_{e\in\Gamma_i}\mathbf{F}_{ie} + \sum_{e\in\Gamma_i}\mathbf{G}^e.\frac{\mathbf{n}_e}{2}$$



$C_i \to$ dual cell
$\mathbf{U}_i \to$ cell average
$\mathbf{F}_{ij}, \mathbf{F}_{ie} \to$ inviscid flux
$\mathbf{G}^T, \mathbf{G}^e \to$ viscous flux

Figure 2: Triangle T and T_e with outward normals

Step 1: Construct a kinetic energy [4] and entropy conservative scheme for the Euler equations using ideas in [2], to satisfy a discrete entropy equality.
Step 2: Add entropy variable based dissipation to obtain a first-order entropy stable scheme (KEPES), satisfying a discrete entropy inequality

$$|C_i|\frac{\mathrm{d}\eta(\mathbf{U}_i)}{\mathrm{d}t} + \sum_{j\in i} q_{ij} \le 0, \qquad q_{ij} = q(\mathbf{U}_i, \mathbf{U}_j, \mathbf{n}_{ij})$$

Step 3: Reconstruct (linear) the **scaled entropy variables** in the dissipation operator while satisfying the sign property, to obtain a second-order entropy stable scheme (KEPES-TeCNO).

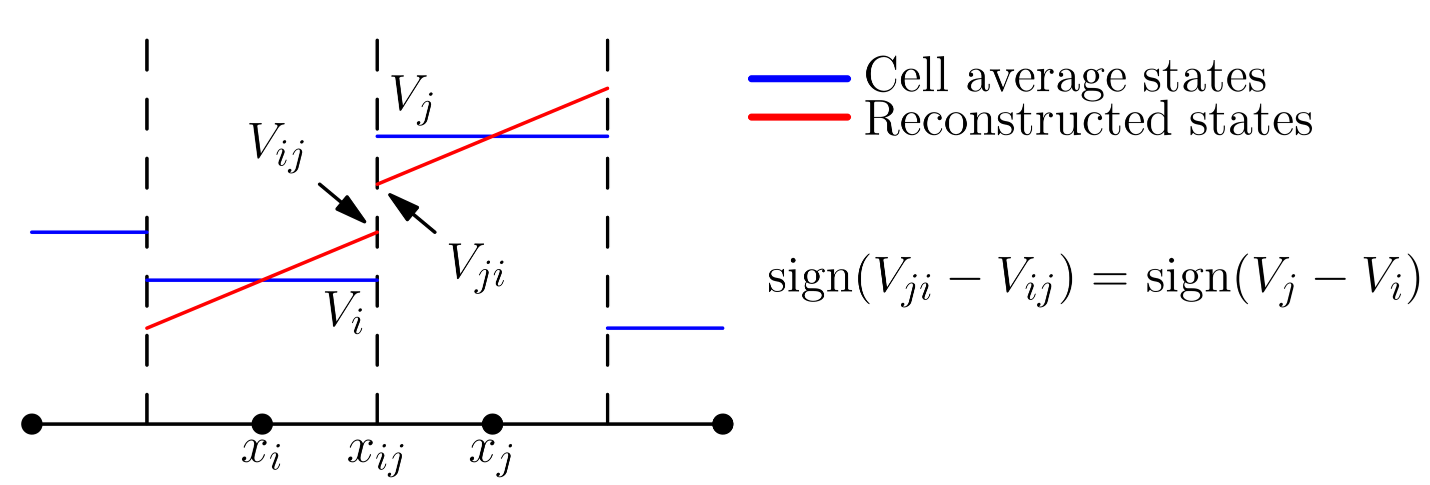


Figure 3: Sign property

Step 4: Approximate the viscous fluxes on the primary cells, taking advantage of the fact that $\mathbf{K} \ge 0$ in (4). This leads to the satisfaction of a discrete analogue of (5). **Step 5:** Integrate the semi-discrete scheme in time using a suitable method.

Modified shock tube problem

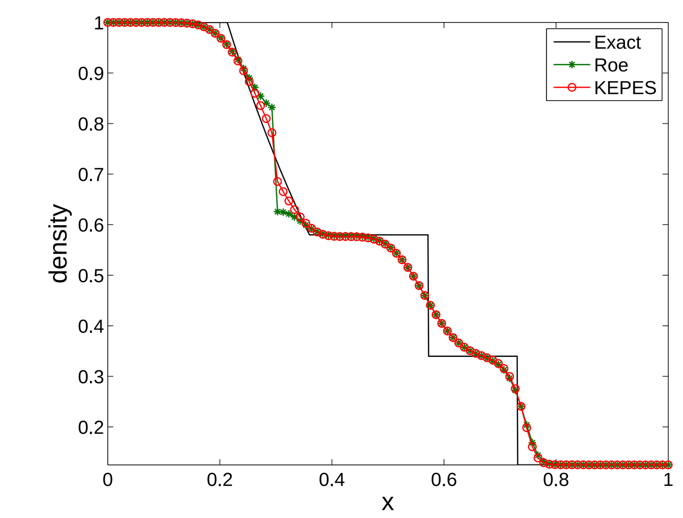


Figure 4: Density plot using Roe and KEPES schemes

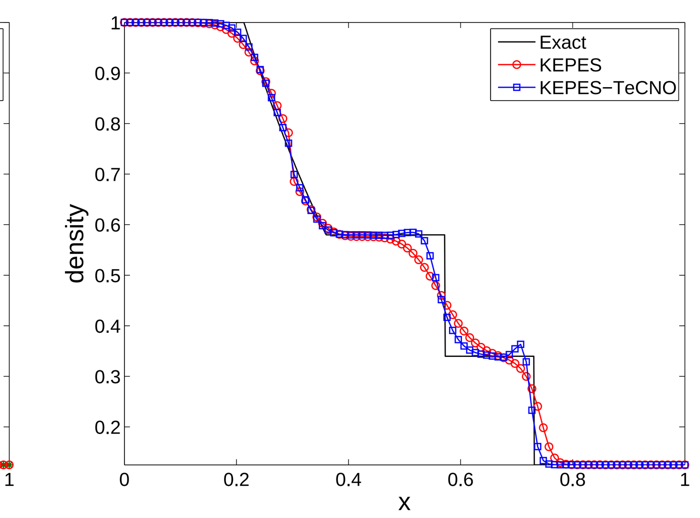


Figure 5: Comparison of KEPES and KEPES-TeCNO

Forward step in wind tunnel at Mach 3

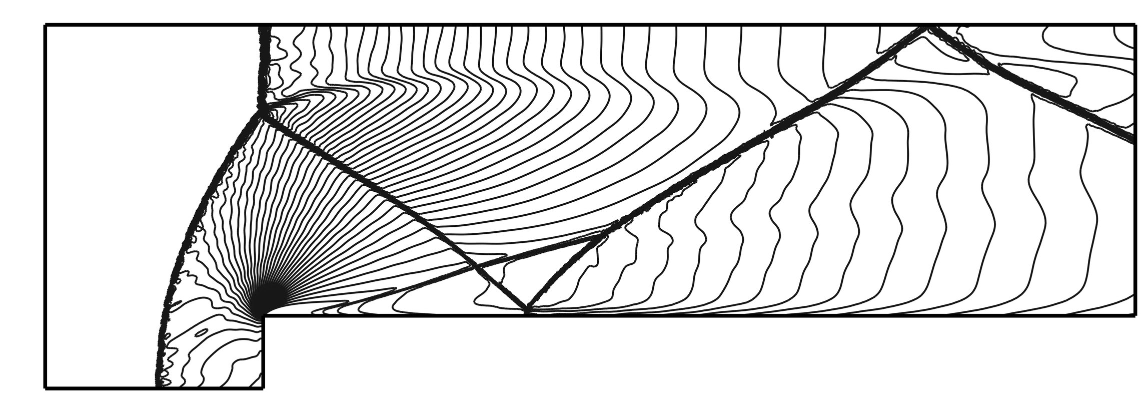


Figure 6: Density contours with KEPES-TeCNO at t=4

Viscous flow past a cylinder, Re=150

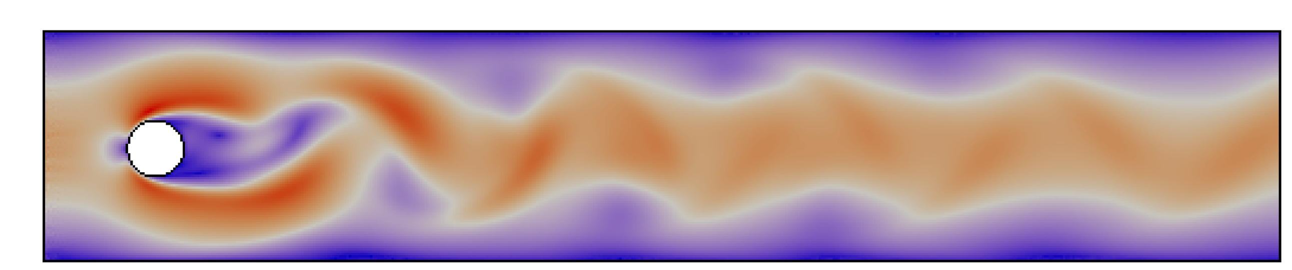


Figure 7: Velocity magnitude plot with entropy conservative scheme and viscous fluxes

Conclusion

A formally second-order entropy stable semi-discrete scheme has been successfully constructed for the Euler equations. The viscous fluxes have been suitably discretised to obtain a scheme that satisfies a discrete global entropy estimate. A few numerical results have been presented to demonstrate the robustness of the schemes. The positive findings pave the way to extend these methods to the three-dimensional setup.

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