A high-resolution energy preserving method for the rotating shallow water equations

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joint work with Praveen Chandrashekar

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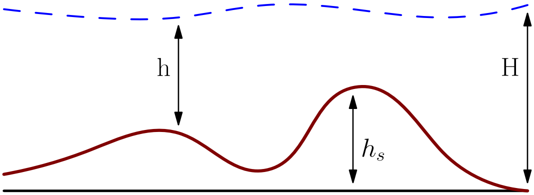


Conservative form:

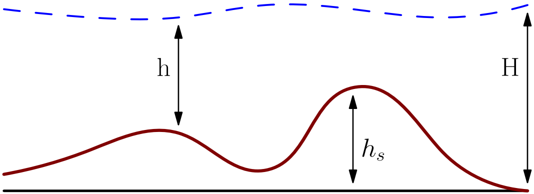
$$\frac{\partial h\mathbf{u}}{\partial t} + \nabla\cdot\left(h\mathbf{u}\otimes\mathbf{u} + \frac{1}{2}gh^2\mathbf{I}\right) = -\mathscr{F}(\mathbf{k}\times h\mathbf{u}) - gh\nabla h_s$$

$$\frac{\partial h}{\partial t} + \nabla\cdot(h\mathbf{u}) = 0$$

$\mathbf{u} = (u_1, u_2) \rightarrow$ horizontal velocity, $g \rightarrow$ acc. due to gravity,

$\mathbf{k} \rightarrow$ unit normal to Earth's surface, $\mathscr{F} = 2\Omega\sin\theta \rightarrow$ Coriolis term

$$\mathbf{k}\cdot\mathbf{u} = 0$$



Vector-invariant form:

$$\frac{\partial \mathbf{u}}{\partial t} + \nabla \Phi + (\omega + \mathscr{F})\mathbf{u}^{\perp} = 0$$

$$\frac{\partial h}{\partial t} + \nabla \cdot (h\mathbf{u}) = 0$$

$$\Phi = gH + K, \qquad K = \frac{1}{2}|\mathbf{u}|^2, \qquad \omega = k \cdot \nabla \times \mathbf{u}, \qquad \mathbf{u}^{\perp} = \mathbf{k} \times \mathbf{u}$$

Conservation of energy

For $\mathbf{U} = (u_1, u_2, h)$, let

$$\eta(\mathbf{U}) = \underbrace{hK + \frac{1}{2}gH^2}_{\text{total energy}}, \quad \mathbf{q} = (gH + K)h\mathbf{u}$$

Using the shallow water equations, we can show

$$\frac{\partial \eta}{\partial t} + \nabla \cdot \mathbf{q} = 0 \quad \Longrightarrow \quad \frac{\mathrm{d}}{\mathrm{d}t} \int \eta \mathrm{d}\mathbf{x} = 0$$

Vorticity dynamics

Absolute vorticity:

$$\vartheta := \omega + \mathscr{F}$$

Taking curl of velocity equations

$$\frac{\partial \vartheta}{\partial t} + \nabla \cdot \vartheta = 0 \quad \Longrightarrow \quad \frac{\mathrm{d}}{\mathrm{d}t} \int \vartheta \mathbf{dx} = 0$$

Potential vorticity:

$$\zeta := \frac{\vartheta}{h}$$

is advected

$$\frac{\partial \zeta}{\partial t} + \mathbf{u} \cdot \nabla \zeta = 0 \quad \Longrightarrow \quad \zeta \text{ constant along stream lines}$$

$$\Longrightarrow \quad \text{max./min. preserved}$$

Other invariants

Every quantity of the form

$$C := \int h\mathcal{G}\left(\frac{\vartheta}{h}\right) \mathbf{dx} \qquad \text{(Casimirs)}$$

is conserved. In particular,

- $\mathcal{G} \equiv 1 \rightarrow$ conservation of total mass
- $\mathcal{G}(s) = s \rightarrow$ conservation of total ϑ
- $\mathcal{G}(s) = s^2/2 \rightarrow$ conservation of total potential enstrophy $\frac{\vartheta^2}{2h}$

Existing energy preserving approaches for SWE

Conservative form: (Entropy framework)

- Energy conservative/stable schemes (Tadmor et al. 2008, 2009)
- Well-balanced energy stable scheme (Fjordholm et al. 2011)
- ...

Vector-invariant form:

- 2nd order energy and potential enstrophy preserving FD scheme (Arakawa and Lamb, 1981)
- Extension to geodesic (icosahedral) grids (Ringler and Randall, 2002)
- TRISK schemes (Thuburn et al) on non-orthogonal polygonal grids using DEC (cannot preserve both simultaneously)
- AL scheme extended to non-orthogonal quad. grids (Toy and Nair, 2016)
- Energy and potential enstrophy preserving scheme using Hamiltonian tools on planar grids (Salmon, 2007)
- Hamiltonian + DEC: extension to arbitrary spherical grids (Eldred and Randall, 2016)

$\mathbf{u} - h - \vartheta$ model

Since ϑ is important, we consider the larger model

$$\begin{aligned}
\frac{\partial \mathbf{u}}{\partial t} + \nabla\Phi + \vartheta\mathbf{u}^{\perp} &= 0 \\
\frac{\partial h}{\partial t} + \nabla\cdot(h\mathbf{u}) &= 0 \\
\frac{\partial \vartheta}{\partial t} + \nabla\cdot(\vartheta\mathbf{u}) &= 0
\end{aligned}$$

Since ϑ is important, we consider the larger model

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{f}_1(\mathbf{U})}{\partial x} + \frac{\partial \mathbf{f}_2(\mathbf{U})}{\partial y} = -\boldsymbol{S},$$
$$\frac{\partial \vartheta}{\partial t} + \frac{\partial J_1}{\partial x} + \frac{\partial J_2}{\partial y} = 0,$$

where

$$\mathbf{U} = \begin{pmatrix} u_1 \\ u_2 \\ h \end{pmatrix}, \quad \mathbf{f}_1(\mathbf{U}) = \begin{pmatrix} \Phi \\ 0 \\ hu_1 \end{pmatrix}, \quad \mathbf{f}_2(\mathbf{U}) = \begin{pmatrix} 0 \\ \Phi \\ hu_2 \end{pmatrix},$$
$$(J_1, J_2) = \vartheta \mathbf{u}^\top, \qquad \boldsymbol{S} = (-u_2\vartheta, u_1\vartheta, 0)^\top$$

Finite difference scheme

Discretize the domain using 2D Cartesian mesh

$$\begin{aligned} I_{i,j} &= [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}) \times [y_{j-\frac{1}{2}}, y_{j+\frac{1}{2}}) \longrightarrow \text{cell} \\ \mathbf{x}_{i,j} &= (x_i, y_j) = (i\Delta x, j\Delta y) \longrightarrow \text{cell center} \end{aligned}$$

Consider the FD scheme

$$\frac{\mathrm{d}\mathbf{U}_{i,j}}{\mathrm{d}t} + \frac{\left(\mathbf{F}^x_{i+\frac{1}{2},j} - \mathbf{F}^x_{i-\frac{1}{2},j}\right)}{\Delta x} + \frac{\left(\mathbf{F}^y_{i,j+\frac{1}{2}} - \mathbf{F}^y_{i,j-\frac{1}{2}}\right)}{\Delta y} = -\boldsymbol{S}_{i,j},$$

$$\frac{\mathrm{d}\vartheta_{i,j}}{\mathrm{d}t} + \frac{\left(J^x_{i+\frac{1}{2},j} - J^x_{i-\frac{1}{2},j}\right)}{\Delta x} + \frac{\left(J^y_{i,j+\frac{1}{2}} - J^y_{i,j-\frac{1}{2}}\right)}{\Delta y} = 0$$

Energy preserving scheme

Theorem (Similar to entropy conservative scheme [Tadmor et al.])

Let $\eta(\mathbf{U})$ and $\mathbf{q} = (q_1, q_2)$ be the energy and energy flux. Define the vector $\mathbf{V} = \eta'(\mathbf{U})$ and the scalar energy potentials

$$\Psi^x := \left\langle \mathbf{V}, \mathbf{f}_1 \right\rangle - q_1, \quad \Psi^y := \left\langle \mathbf{V}, \mathbf{f}_2 \right\rangle - q_2$$

If the numerical flux functions $\mathbf{F}^x, \mathbf{F}^y$ satisfy the conditions

$$\left\langle \Delta \mathbf{V}_{i+\frac{1}{2},j}, \mathbf{F}^x_{i+\frac{1}{2},j} \right\rangle = \Delta \Psi^x_{i+\frac{1}{2},j}, \quad \left\langle \Delta \mathbf{V}_{i,j+\frac{1}{2}}, \mathbf{F}^y_{i,j+\frac{1}{2}} \right\rangle = \Delta \Psi^y_{i,j+\frac{1}{2}},$$

then, the scheme preserves energy, i.e.,

$$\frac{\mathrm{d}\eta(\mathbf{U}_i)}{\mathrm{d}t} + \frac{\left(q^x_{i+\frac{1}{2},j} - q^x_{i-\frac{1}{2},j}\right)}{\Delta x} + \frac{\left(q^y_{i,j+\frac{1}{2}} - q^y_{i,j-\frac{1}{2}}\right)}{\Delta y} = 0.$$

The following second-order two-point flux satisfies the algebraic condition

$$\mathbf{F}^x_{i+\frac{1}{2},j} = \frac{1}{2}\big(\mathbf{f}_1(\mathbf{U}_{i,j}) + \mathbf{f}_1(\mathbf{U}_{i+1,j})\big), \quad \mathbf{F}^y_{i,j+\frac{1}{2}} = \frac{1}{2}\big(\mathbf{f}_2(\mathbf{U}_{i,j}) + \mathbf{f}_2(\mathbf{U}_{i,j+1})\big)$$

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Can obtain $2p$-th order flux using

$$\mathbf{F}^{x,2p}_{i+\frac{1}{2}} = \sum_{r=1}^{p} \alpha_r^p \sum_{s=0}^{r-1} \mathbf{F}^x(\mathbf{U}_{i-s}, \mathbf{U}_{i-s+r}) \quad \text{(Lefloch et al., 2002)}$$

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- What about the flux for ϑ?
- Source term depends on ϑ.
- Central scheme bad near steep gradients.

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Use finite difference WENO for ϑ (Shu, 1998)
Step 1 (Flux splitting):

$$(J_1)^{\mathscr{U}}(\vartheta, \mathbf{u}) = \frac{1}{2}\left(J_1(\vartheta) + \alpha^x \vartheta\right), \quad (J_1)^{\mathscr{D}}(\vartheta, \mathbf{u}) = \frac{1}{2}\left(J_1(\vartheta) - \alpha^x \vartheta\right)$$

where $\alpha^x = \max_{i,j}\{|(u_1)_{i,j}|\}$

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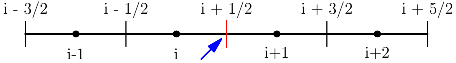
where $\alpha^x = \max_{i,j}\{|(u_1)_{i,j}|\}$

Step 2 (Reconstruct upwind flux): Set $\phi_i = (J_1)^{\mathscr{U}}(\vartheta_i, \mathbf{u}_i)$

$$\phi^{(0),-}_{i+\frac{1}{2}} = \frac{1}{3}\phi_i + \frac{5}{6}\phi_{i+1} - \frac{1}{6}\phi_{i+2}$$

$$\phi^{(1),-}_{i+\frac{1}{2}} = -\frac{1}{6}\phi_{i-1} + \frac{5}{6}\phi_i + \frac{1}{3}\phi_{i+1}$$

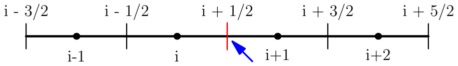
$$\phi^{(2),-}_{i+\frac{1}{2}} = \frac{1}{3}\phi_{i-2} - \frac{7}{6}\phi_{i-1} + \frac{11}{6}\phi_i$$



5-th order WENO approximation is given by

$$(J^x)^{\mathscr{U}}_{i+\frac{1}{2}} = \phi^-_{i+\frac{1}{2}} = w_0\phi^{(0),-}_{i+\frac{1}{2}} + w_1\phi^{(1),-}_{i+\frac{1}{2}} + w_2\phi^{(2),-}_{i+\frac{1}{2}}$$

Use finite difference WENO for ϑ (Shu, 1998)
Step 3 (Reconstruct downwind flux): Set $\phi_{i+k} = (J_1)^{\mathscr{D}}(\vartheta_{i-k+1}, \mathbf{u}_{i-k+1})$ and repeat Step 2 to get $(J^x)^{\mathscr{D}}_{i+\frac{1}{2}}$



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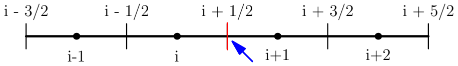

Step 4: Add fluxes to get

$$J^x_{i+\frac{1}{2}} = (J^x)^{\mathscr{U}}_{i+\frac{1}{2}} + (J^x)^{\mathscr{D}}_{i+\frac{1}{2}}$$

Use finite difference WENO for ϑ (Shu, 1998)
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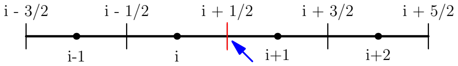

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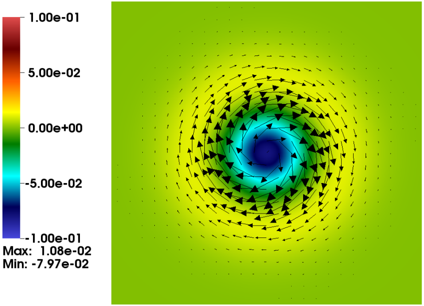
VI-EP4 scheme:
- 4-th order energy preserving fluxes $\mathbf{F}^{x,4}_{i+\frac{1}{2}}$, $\mathbf{F}^{y,4}_{j+\frac{1}{2}}$
- 5-th order FD-WENO for ϑ
- time-marching with RK4

Advecting vortex: accuracy

$$u_1 = 1 + c_1 y \exp\left(-c_2 r^2\right)\big), \quad u_2 = -c_1 x \exp\left(-c_2 r^2\right)\big),$$

$$H = 1 - \frac{c_1^2}{4c_2 g} \exp\left(-2c_2 r^2\right)\big), \quad h_s = 0, \quad r = \sqrt{x^2 + y^2},$$

$$c_1 = 0.04, \quad c_2 = 0.02, \quad g = 1, \quad \mathscr{F} = 0.$$



Advecting vortex: Accuracy

N	u_1				u_2			
	L_h^1		L_h^∞		L_h^1		L_h^∞	
	error	rate	error	rate	error	rate	error	rate
50	1.43e-00	-	5.31e-03	-	1.87e-00	-	9.59e-03	-
100	5.05e-02	4.83	3.30e-04	4.01	6.84e-02	4.77	4.69e-04	4.35
200	1.63e-03	4.95	1.11e-05	4.90	2.27e-03	4.91	1.60e-05	4.87
300	2.23e-04	4.91	1.40e-06	5.11	3.14e-04	4.88	2.11e-06	4.99
400	5.62e-05	4.79	3.12e-07	5.21	7.96e-05	4.77	4.99e-07	5.01

N	h				ϑ			
	L_h^1		L_h^∞		L_h^1		L_h^∞	
	error	rate	error	rate	error	rate	error	rate
50	7.59e-01	-	2.11e-03	-	6.77e-01	-	6.14e-03	-
100	2.78e-02	4.77	9.14e-05	4.53	3.55e-02	4.25	3.04e-04	4.33
200	1.13e-03	4.62	3.55e-06	4.68	1.14e-03	4.96	1.18e-05	4.68
300	1.88e-04	4.42	5.57e-07	4.57	1.34e-04	5.27	1.65e-06	4.85
400	5.47e-05	4.29	1.55e-07	4.42	2.96e-05	5.26	3.99e-07	4.95

Perturbed geostrophic balance

Domain: $[-5,5]^2$, $T_f = 50$. Geostrophic balance:

$$\mathbf{u} = \left(-\frac{g}{\mathscr{F}}\frac{\mathsf{d}h_0}{\mathsf{d}y}, 0\right), \quad \omega = \frac{g}{\mathscr{F}}\frac{\mathsf{d}^2 h_0}{\mathsf{d}y^2}$$

$$h_0(y) = 1 + 0.1\left[\tanh\left(\frac{1-y^2}{0.5}\right) + 1\right], \quad g = 1, \quad \mathscr{F} = 1$$

Perturb:

$$h(x,y) = h_0(y) + h_\delta \exp(-r^2), \quad r^2 = x^2 + y^2, \quad h_\delta = 5e-3$$

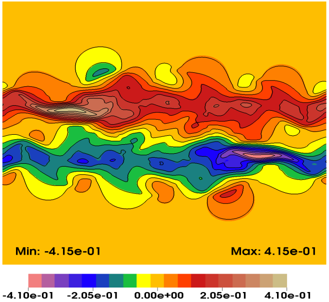
Vector-Invariant model: Solves $\mathbf{u}$, h, ϑ

- VI-EP4

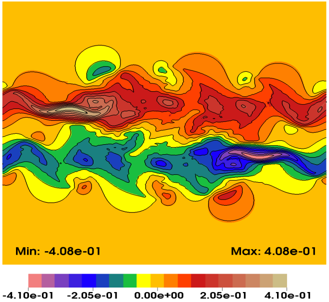
Conservative model: Solves $h\mathbf{u}$, h

- CM-EP4 (Fjordholm et al., 2009)
- CM-WENO5
- Vorticity with fourth order approximation from $\mathbf{u}$

Perturbed geostrophic balance (u_1 profiles)



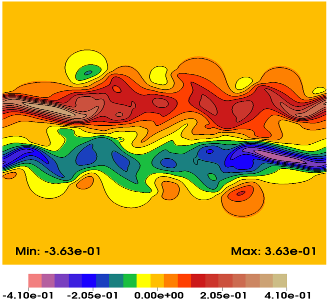
VI-EP4



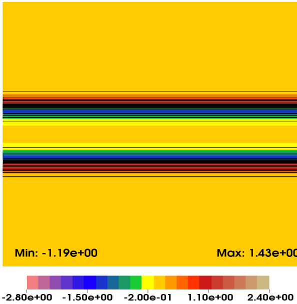
CM-EP4

(has implicit dissipation)

CM-WENO5



Perturbed geostrophic balance (ω profiles)



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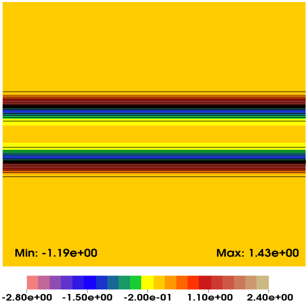
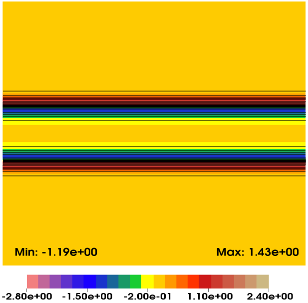


CM-EP4



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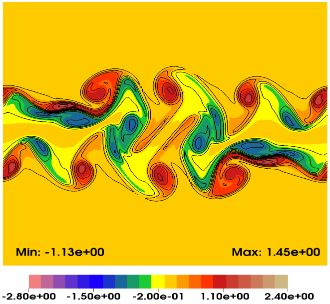

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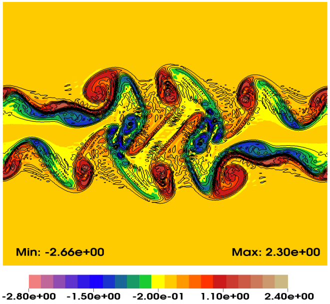
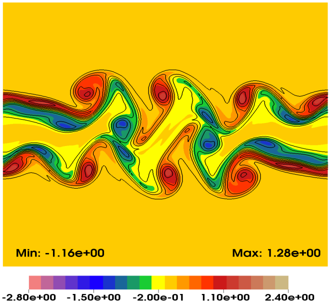


CM-EP4



CM-WENO5

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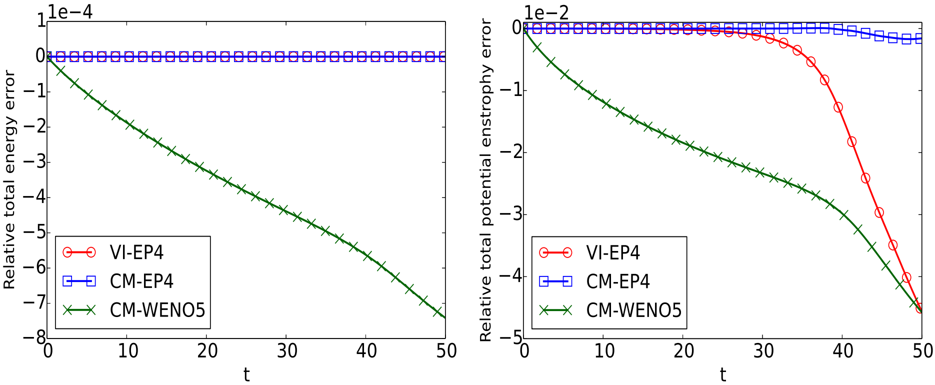

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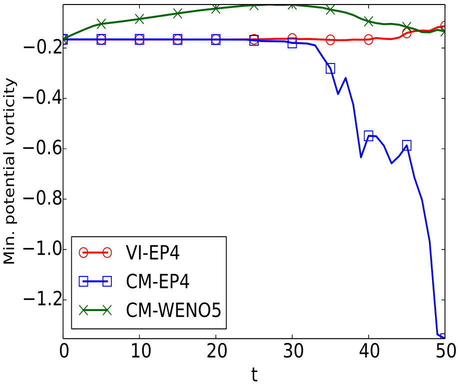
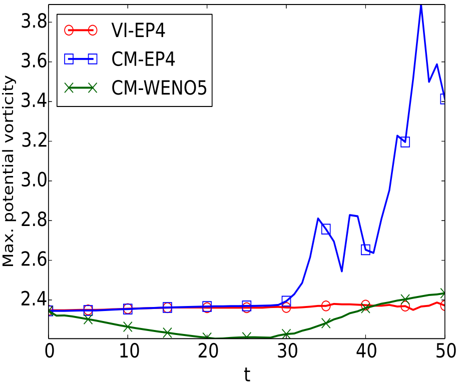




Flow over an isolated mountain (Toy et al. 2016)

Initially designed for simulation on a sphere.
Domain: $[-\pi a, \pi a]^2$, $a = 6.37 \times 10^6$m $\longrightarrow$ flattened Earth

Geostrophic balance:

$$\mathbf{u}(y) = \left(u_0 \cos\left(\frac{y}{a}\right), 0\right), \quad \mathscr{F}(y) = 2\Omega \sin\left(\frac{y}{a}\right)$$
$$H(y) = h_0 - \frac{a}{g}\Omega u_0 \sin^2\left(\frac{y}{a}\right)$$

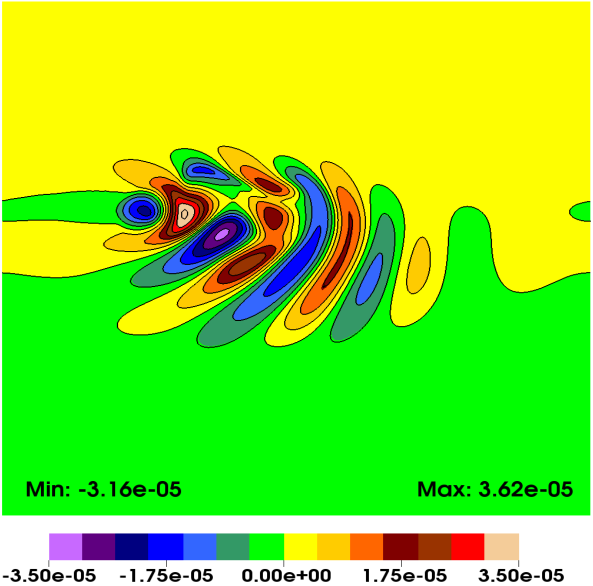
where

$$h_0 = 5960\text{m}, \quad u_0 = 20\text{ms}^{-1}, \quad \Omega = 7.292 \times 10^{-5}\text{s}^{-1}$$

Introduce a mountain:

$$h_s(x, y) = h_{s,\delta}\left(1 - \frac{r}{R}\right), \quad r = \min(R, \sqrt{(x - x_c)^2 + (y - y_c)^2},$$
$$h_{s,\delta} = 2000\text{m}, \quad (x_c, y_c) = \left(-\frac{\pi a}{2}, \frac{\pi a}{6}\right), \quad R = \frac{\pi a}{9}$$

Flow over an isolated mountain ($T = 7$ days)



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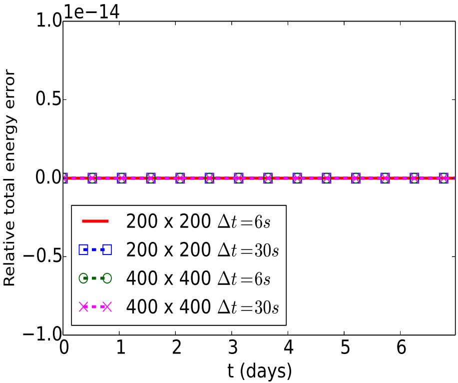

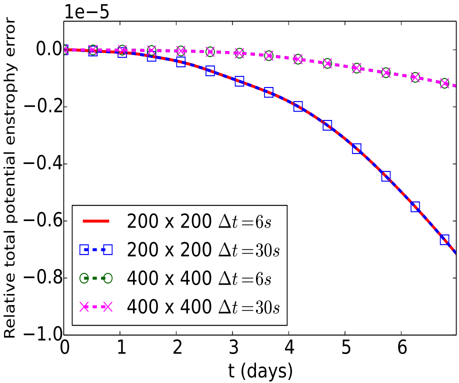

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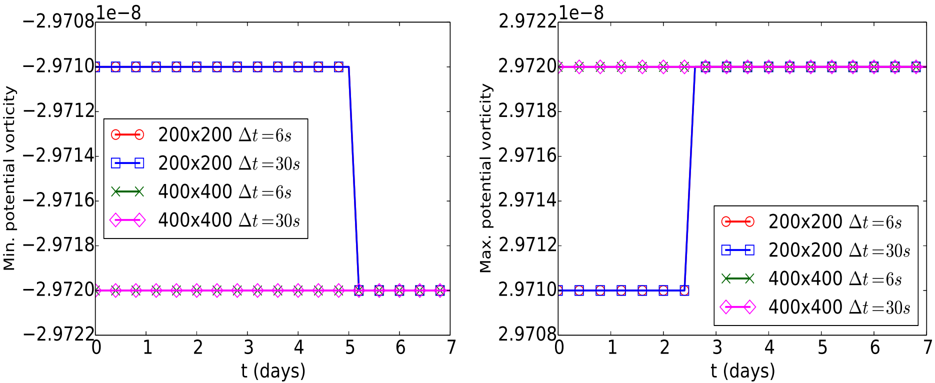
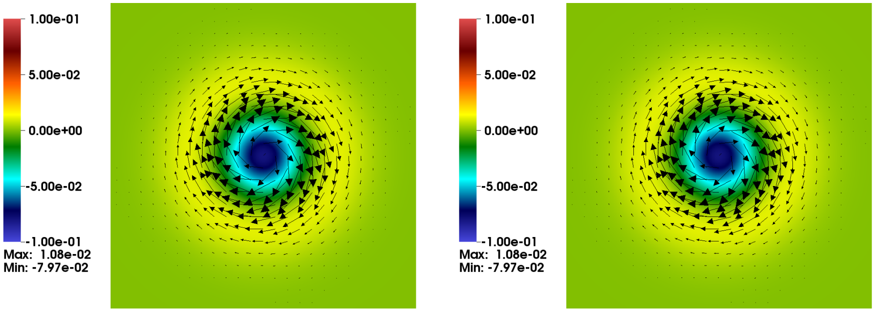


Conclusion

- Constructed fourth-order accurate energy preserving scheme for rSWE
- Implicit dissipation through upwinding in ϑ
- Smoother approximation of ϑ (via solver)
- Arbitrary high-order energy preserving scheme can be constructed

Thank you.

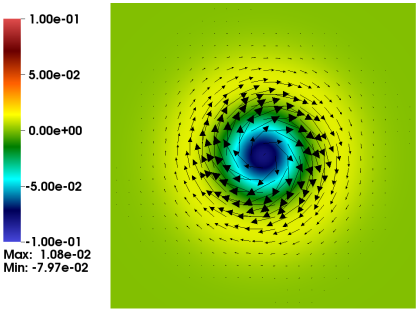
Advecting vortex: Effect of Coriolis force



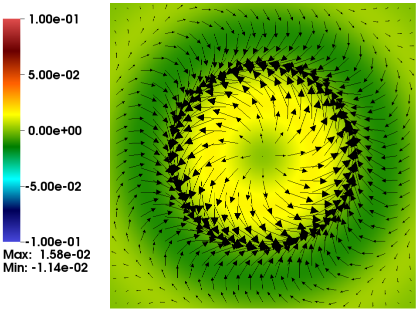
$\mathscr{F} = 0$ $\mathscr{F} = 1$

Domain: $[-40, 40]^2$, Mesh: 200×200.

Advecting vortex: Effect of Coriolis force



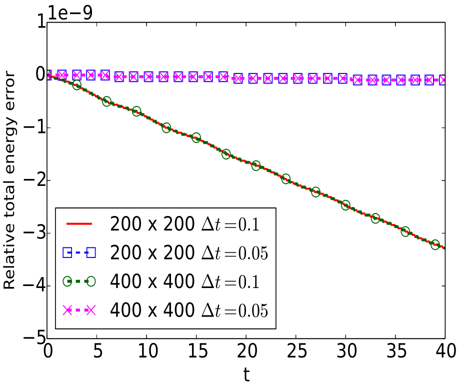
$\mathscr{F} = 0$



$\mathscr{F} = 1$

Domain: $[-40, 40]^2$, Mesh: 200×200.

Effect of Coriolis terms



Perturbed geostrophic balance