A third-order entropy stable scheme for the compressible Euler equations

Deep Ray

Tata Institute of Fundamental Research
Centre for Applicable Mathematics
Bangalore
deep@math.tifrbng.res.in
http://math.tifrbng.res.in/~deep

HYP2016, Aachen

Work done with **Ulrik S. Fjordholm**, NTNU, Trondheim

D.R. supported in part by AIRBUS Group Corporate Foundation Chair in Mathematics of Complex Systems, established in TIFR/ICTS, Bangalore

Conservation laws and entropy conditions

Consider the Cauchy problem

$$\begin{aligned} \partial_t \mathbf{U} + \nabla.\mathbf{f}(\mathbf{U}) = 0 \qquad & \forall \quad (\mathbf{x}, t) \in \mathbb{R}^d \times \mathbb{R}^+ \\ \mathbf{U}(\mathbf{x}, 0) = \mathbf{U}_0(\mathbf{x}) \qquad & \forall \quad \mathbf{x} \in \mathbb{R}^d \end{aligned}$$

where $\mathbf{f}(\mathbf{U}) = (\mathbf{f}_1(\mathbf{U}), ..., \mathbf{f}_d(\mathbf{U}))$

Entropy-entropy flux pair $(\eta(\mathbf{U}), \mathbf{q}(\mathbf{U}))$ with

$$q_i' = \mathbf{V}^\top \partial_{\mathbf{U}} \mathbf{f}_i', \quad i = 1, ..., d$$

$$\mathbf{V}(\mathbf{U}) = \eta'(\mathbf{U}) \rightarrow \text{entropy variables}$$

For discontinuous solutions we have

$$\boxed{\partial_t \eta(\mathbf{U}) + \nabla.\mathbf{q}(\mathbf{U}) \leq 0}$$

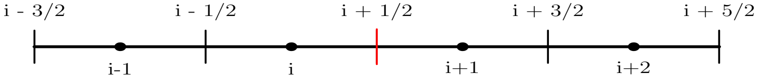
Existence, uniqueness of solutions for scalar conservation laws (Kruzkov).

AIM: Design high-order numerical schemes satisfying a discrete entropy inequality.

Outline

- Finite difference scheme
- Entropy conservative/stable schemes (Tadmor)
- Higher-order entropy stable schemes
- Sign preservation and other essentials
- SP-WENO
- A correction for systems
- Conclusion and future scope

Finite difference scheme



Discretize the domain $\Omega = \bigcup_i I_i$, where

$$I_i = [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}), \qquad x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}} \equiv h$$

Consider the semi-discrete finite difference scheme

$$\frac{\mathrm{d}\mathbf{U}_i}{\mathrm{d}t} + \frac{1}{h}\left(\mathbf{F}_{i+\frac{1}{2}} - \mathbf{F}_{i-\frac{1}{2}}\right) = 0$$

$\mathbf{F}_{i+\frac{1}{2}}(t) = \mathbf{F}(\mathbf{U}_i(t), \mathbf{U}_{i+1}(t))$ is the numerical flux satisfying

1. Consistency: $\mathbf{F}(\mathbf{U}, \mathbf{U}) = \mathbf{f}(\mathbf{U})$
2. Conservation: $\mathbf{F}(\mathbf{U}_1, \mathbf{U}_2) = \mathbf{F}(\mathbf{U}_2, \mathbf{U}_1)$

Constructing an entropy stable scheme

Step 1: Entropy conservation

Entropy conservative scheme

A scheme is entropy conservative if

$$\frac{\mathrm{d}\eta(\mathbf{U}_i)}{\mathrm{d}t} + \frac{1}{h}\left(q^*_{i+\frac{1}{2}} - q^*_{i-\frac{1}{2}}\right) = 0$$

where $q^*_{i+\frac{1}{2}}$ is a consistent numerical entropy flux.

Constructing an entropy stable scheme

Step 1: Entropy conservation

Entropy conservative scheme

A scheme is entropy conservative if

$$\frac{\mathrm{d}\eta(\mathbf{U}_i)}{\mathrm{d}t} + \frac{1}{h}\left(q^*_{i+\frac{1}{2}} - q^*_{i-\frac{1}{2}}\right) = 0$$

where $q^*_{i+\frac{1}{2}}$ is a consistent numerical entropy flux.

Notation: $\Delta(\cdot)_{i+\frac{1}{2}} = (\cdot)_{i+1} - (\cdot)_i$

A sufficient condition (Tadmor)

A scheme with flux $\mathbf{F}^*_{i+\frac{1}{2}}$ is entropy conservative if

$$\Delta\mathbf{V}^\top_{i+\frac{1}{2}}\mathbf{F}^*_{i+\frac{1}{2}} = \Delta\Psi_{i+\frac{1}{2}}$$

where $\Psi(\mathbf{U}) := \mathbf{V}^\top\mathbf{f} - q$ is the entropy potential.

Remarks

- Unique entropy conservative flux for scalar equations (given η)

$$F^*_{i+\frac{1}{2}} = \frac{\Delta \Psi_{i+\frac{1}{2}}}{\Delta v_{i+\frac{1}{2}}}.$$

- Higher order entropy conservative schemes (Mercier et al.)

$$\mathbf{F}^{*,2p}_{i+\frac{1}{2}} = \sum_{r=1}^{p} \alpha_r^p \sum_{s=0}^{r-1} \mathbf{F}^*(\mathbf{U}_{i-s}, \mathbf{U}_{i-s+r}) \qquad (2p\text{-th order})$$

- Fourth order entropy conservative flux

$$\mathbf{F}^{*,4} = \frac{4}{3}\mathbf{F}^*(\mathbf{U}_i, \mathbf{U}_{i+1}) - \frac{1}{6}\Big(\mathbf{F}^*(\mathbf{U}_{i-1}, \mathbf{U}_{i+1}) + \mathbf{F}^*(\mathbf{U}_i, \mathbf{U}_{i+2})\Big)$$

Constructing an entropy stable scheme

Step 2: Add dissipation
Entropy dissipated near discontinuities.

$$\mathbf{F}_{i+\frac{1}{2}} = \mathbf{F}^{*,2p}_{i+\frac{1}{2}} - \frac{1}{2}\mathbf{D}_{i+\frac{1}{2}}\Delta\mathbf{V}_{i+\frac{1}{2}}, \quad \mathbf{D}_{i+\frac{1}{2}} = \mathbf{D}^{\top}_{i+\frac{1}{2}} \geq 0$$

Entropy stable scheme

The scheme with the above flux satisfies

$$\frac{\mathrm{d}\eta(\mathbf{U}_i)}{\mathrm{d}t} + \frac{1}{h}\left(q_{i+\frac{1}{2}} - q_{i-\frac{1}{2}}\right) \leq 0$$

Scalar: $\mathbf{D}_{i+\frac{1}{2}} = d_{i+\frac{1}{2}} \approx |f'(u)|$
System: $\mathbf{D}_{i+\frac{1}{2}} = \mathbf{R}_{i+\frac{1}{2}}\Lambda_{i+\frac{1}{2}}\mathbf{R}^{\top}_{i+\frac{1}{2}}$

High-order diffusion

$$\Delta \mathbf{V}_{i+\frac{1}{2}} \sim \mathcal{O}(h) \quad \Longrightarrow \quad \text{first order flux}$$

High-order diffusion

$$\Delta \mathbf{V}_{i+\frac{1}{2}} \sim \mathcal{O}(h) \quad \Longrightarrow \quad \text{first order flux}$$

Idea: At each $x_{i+\frac{1}{2}}$, reconstruct each component v in I_i, I_{i+1} with polynomials $v_i(x), v_{i+1}(x)$ respectively.

$$v^-_{i+\frac{1}{2}} = v_i(x_{i+\frac{1}{2}}), \qquad v^+_{i+\frac{1}{2}} = v_{i+1}(x_{i+\frac{1}{2}}), \qquad [\![v]\!]_{i+\frac{1}{2}} = v^+_{i+\frac{1}{2}} - v^-_{i+\frac{1}{2}}$$

Replace $\Delta v_{i+\frac{1}{2}}$ by $[\![v]\!]_{i+\frac{1}{2}} \sim \mathcal{O}(h^k)$.

High-order diffusion

$$\Delta \mathbf{V}_{i+\frac{1}{2}} \sim \mathcal{O}(h) \quad \Longrightarrow \quad \text{first order flux}$$

Idea: At each $x_{i+\frac{1}{2}}$, reconstruct each component v in I_i, I_{i+1} with polynomials $v_i(x), v_{i+1}(x)$ respectively.

$$v^-_{i+\frac{1}{2}} = v_i(x_{i+\frac{1}{2}}), \qquad v^+_{i+\frac{1}{2}} = v_{i+1}(x_{i+\frac{1}{2}}), \qquad [\![v]\!]_{i+\frac{1}{2}} = v^+_{i+\frac{1}{2}} - v^-_{i+\frac{1}{2}}$$

Replace $\Delta v_{i+\frac{1}{2}}$ by $[\![v]\!]_{i+\frac{1}{2}} \sim \mathcal{O}(h^k)$.

How do we ensure entropy stability?

High-order diffusion

Define scaled entropy variables $\mathbf{Z} = \mathbf{R}_{i+\frac{1}{2}}^{\top}\mathbf{V}$

$$\mathbf{F}_{i+\frac{1}{2}} = \mathbf{F}^{*}_{i+\frac{1}{2}} - \frac{1}{2}\mathbf{R}_{i+\frac{1}{2}}\Lambda_{i+\frac{1}{2}}\Delta\mathbf{Z}_{i+\frac{1}{2}}$$

We reconstruct scaled entropy variables

High-order diffusion

Define scaled entropy variables $\mathbf{Z} = \mathbf{R}_{i+\frac{1}{2}}^{\top}\mathbf{V}$

$$\mathbf{F}_{i+\frac{1}{2}} = \mathbf{F}^{*}_{i+\frac{1}{2}} - \frac{1}{2}\mathbf{R}_{i+\frac{1}{2}}\Lambda_{i+\frac{1}{2}}\Delta\mathbf{Z}_{i+\frac{1}{2}}$$

We reconstruct scaled entropy variables

Sign property (Fjordholm et al.)

The scheme with the numerical flux

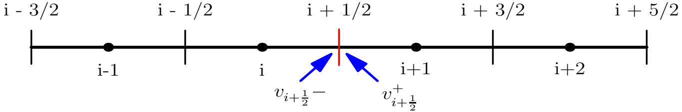
$$\mathbf{F}_{i+\frac{1}{2}} = \mathbf{F}^{*}_{i+\frac{1}{2}} - \frac{1}{2}\mathbf{R}_{i+\frac{1}{2}}\Lambda_{i+\frac{1}{2}}[\![\mathbf{Z}]\!]_{i+\frac{1}{2}}$$

is entropy stable if the following sign property holds (component-wise) for $x_{i+\frac{1}{2}}$

$$\boxed{\mathrm{sign}([\![z]\!]_{i+\frac{1}{2}}) = \mathrm{sign}(\Delta z_{i+\frac{1}{2}}).}$$

For scalar equations, we work directly with entropy variables

Second order reconstruction with minmod limiter



$$
\begin{aligned}
v^-_{i+\frac{1}{2}} &= v_i + \frac{1}{2}\mathsf{minmod}(\Delta v_{i+\frac{1}{2}}, \Delta v_{i-\frac{1}{2}}) \\
v^+_{i+\frac{1}{2}} &= v_{i+1} - \frac{1}{2}\mathsf{minmod}(\Delta v_{i+\frac{3}{2}}, \Delta v_{i+\frac{1}{2}})
\end{aligned}
$$

where

$$
\mathsf{minmod}(a,b) = \begin{cases} \operatorname{sign}(a)\min\left(|a|,|b|\right), & \text{if } \operatorname{sign}(a) = \operatorname{sign}(b) \\ 0, & \text{o.w.} \end{cases}
$$

ENO interpolation (Fjordholm et. al)
Construct k-th degree polynomial using adaptive stencil.

Third order reconstruction (Cheng and Nie)
Quadratic polynomial + limiter.

WENO reconstruction

Idea: Convex combination of lower order polynomials to get higher order accuracy: $k \to (2k-1)$

Advantages:

- Full utilization of (2k-1) cells
- ENO has accuracy issues due to unstable stencils [1]

[2] A. M. Rogerson and E. Meiburg *A numerical study of the convergence properties of ENO schemes.* (1990)

WENO reconstruction

Idea: Convex combination of lower order polynomials to get higher order accuracy: $k \to (2k-1)$

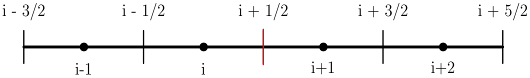
Advantages:

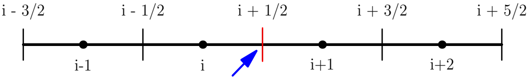
- Full utilization of (2k-1) cells
- ENO has accuracy issues due to unstable stencils [1]

Choose weights to satisfy the sign property?

[2] A. M. Rogerson and E. Meiburg *A numerical study of the convergence properties of ENO schemes.* (1990)

WENO-3



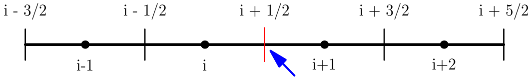


Reconstruction from left: $v^{-}_{i+\frac{1}{2}}$

$$S_0 = \{x_i, x_{i+1}\}, \quad S_1 = \{x_{i-1}, x_i\}$$

Second order reconstructions are:

$$v^{(0),-}_{i+\frac{1}{2}} = \frac{v_i}{2} + \frac{v_{i+1}}{2}, \qquad v^{(1),-}_{i+\frac{1}{2}} = -\frac{v_{i-1}}{2} + \frac{3v_i}{2}$$



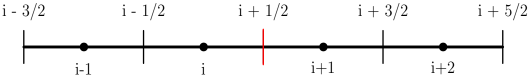
Reconstruction from right: $v^{+}_{i+\frac{1}{2}}$

$$S_0 = \{x_{i+1}, x_{i+2}\}, \quad S_1 = \{x_i, x_{i+1}\}$$

Second order reconstructions are:

$$v^{(0),+}_{i+\frac{1}{2}} = \frac{3v_{i+1}}{2} - \frac{v_{i+2}}{2}, \qquad v^{(1),+}_{i+\frac{1}{2}} = \frac{v_i}{2} + \frac{v_{i+1}}{2}$$

WENO-3



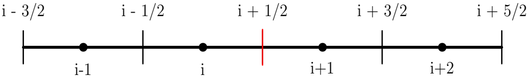
WENO-3: convex combination of ENO-2 polynomials

$$v^-_{i+\frac{1}{2}} = w_0 v^{(0),-}_{i+\frac{1}{2}} + w_1 v^{(1),-}_{i+\frac{1}{2}}, \qquad v^+_{i+\frac{1}{2}} = \tilde{w}_0 v^{(0),+}_{i+\frac{1}{2}} + \tilde{w}_1 v^{(1),+}_{i+\frac{1}{2}}$$

Taylor expansion gives the (order) constraints

$$\begin{aligned} w_0 + w_1 &= 1 \\ \frac{w_0}{8} - \frac{3w_1}{8} &= C_1 = \mathcal{O}(h) \end{aligned} \qquad \begin{aligned} \widetilde{w}_0 + \widetilde{w}_1 &= 1 \\ -\frac{\widetilde{3w_0}}{8} + \frac{\widetilde{w}_1}{8} &= C_2 = \mathcal{O}(h) \end{aligned}$$

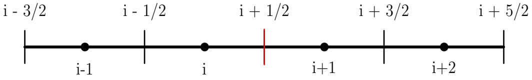
WENO-3



WENO-3: convex combination of ENO-2 polynomials

$$v^-_{i+\frac{1}{2}} = w_0 v^{(0),-}_{i+\frac{1}{2}} + w_1 v^{(1),-}_{i+\frac{1}{2}}, \qquad v^+_{i+\frac{1}{2}} = \tilde{w}_0 v^{(0),+}_{i+\frac{1}{2}} + \tilde{w}_1 v^{(1),+}_{i+\frac{1}{2}}$$

$$\boxed{\begin{aligned} w_0 &= \frac{3}{4} + 2C_1 & \qquad w_1 &= \frac{1}{4} - 2C_1 \\ \widetilde{w}_0 &= \frac{1}{4} - 2C_2 & \qquad \widetilde{w}_1 &= \frac{3}{4} + 2C_2 \end{aligned}}$$

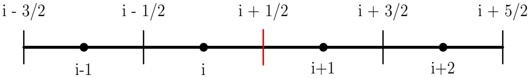


WENO-3: convex combination of ENO-2 polynomials

$$v^-_{i+\frac{1}{2}} = w_0 v^{(0),-}_{i+\frac{1}{2}} + w_1 v^{(1),-}_{i+\frac{1}{2}}, \qquad v^+_{i+\frac{1}{2}} = \tilde{w}_0 v^{(0),+}_{i+\frac{1}{2}} + \tilde{w}_1 v^{(1),+}_{i+\frac{1}{2}}$$

$$\boxed{\begin{array}{ll} w_0 = \dfrac{3}{4} + 2C_1 & \quad w_1 = \dfrac{1}{4} - 2C_1 \\ \widetilde{w}_0 = \dfrac{1}{4} - 2C_2 & \quad \widetilde{w}_1 = \dfrac{3}{4} + 2C_2 \end{array}}$$

"Ideal" weights



WENO-3: convex combination of ENO-2 polynomials

$$v^-_{i+\frac{1}{2}} = w_0 v^{(0),-}_{i+\frac{1}{2}} + w_1 v^{(1),-}_{i+\frac{1}{2}}, \qquad v^+_{i+\frac{1}{2}} = \tilde{w}_0 v^{(0),+}_{i+\frac{1}{2}} + \tilde{w}_1 v^{(1),+}_{i+\frac{1}{2}}$$

$$\boxed{\begin{array}{ll} w_0 = \dfrac{3}{4} + 2C_1 & w_1 = \dfrac{1}{4} - 2C_1 \\ \widetilde{w}_0 = \dfrac{1}{4} - 2C_2 & \widetilde{w}_1 = \dfrac{3}{4} + 2C_2 \end{array}}$$

"Ideal" weights

Issues:

- No smoothness indicators
- Sign property

Essential properties

Consistency:

$$0 \le w_0, w_1, \widetilde{w}_0, \widetilde{w}_1 \le 1$$

or

$$-\frac{3}{8} \le C_1, C_2 \le \frac{1}{8}$$

Essential properties

Consistency:

$$0 \leq w_0, w_1, \widetilde{w}_0, \widetilde{w}_1 \leq 1$$

or

$$-\frac{3}{8} \leq C_1, C_2 \leq \frac{1}{8}$$

Sign property:

$$\mathsf{sign}(\Delta v_{i+\frac{1}{2}}) = \mathsf{sign}([\![v]\!]_{i+\frac{1}{2}})$$

Define,

$$\theta_i^- = \frac{\Delta v_{i+\frac{1}{2}}}{\Delta v_{i-\frac{1}{2}}}, \qquad \theta_i^+ = \frac{1}{\theta_i^-} = \frac{\Delta v_{i-\frac{1}{2}}}{\Delta v_{i+\frac{1}{2}}}$$

Thus,

$$[\![v]\!]_{i+\frac{1}{2}} = \frac{1}{2}\left[\widetilde{w}_0(1-\theta_{i+1}^-) + w_1(1-\theta_i^+)\right]\Delta v_{i+\frac{1}{2}}$$

Need,

$$\left[\widetilde{w}_0(1-\theta_{i+1}^-) + w_1(1-\theta_i^+)\right] \geq 0$$

Essential properties

Negation symmetry: No bias towards positive or negative values.
Under the transform $v \mapsto -v$

$$\Delta v_{j+\frac{1}{2}} \mapsto -\Delta v_{j+\frac{1}{2}} \qquad \forall\ j \in \mathbb{Z}$$

But θ_j^- or θ_j^+ remain unchanged.
A sufficient condition

$$C_1 = C_1(\theta_i^+, \theta_{i+1}^-), \qquad C_2 = C_2(\theta_i^+, \theta_{i+1}^-)$$

Essential properties

Negation symmetry: No bias towards positive or negative values.
Under the transform $v \mapsto -v$

$$\Delta v_{j+\frac{1}{2}} \mapsto -\Delta v_{j+\frac{1}{2}} \qquad \forall\, j \in \mathbb{Z}$$

But θ_j^- or θ_j^+ remain unchanged.
A sufficient condition

$$C_1 = C_1(\theta_i^+, \theta_{i+1}^-), \qquad C_2 = C_2(\theta_i^+, \theta_{i+1}^-)$$

Mirror property: Mirroring solution about $x_{i+\frac{1}{2}}$ gives

$$\theta_{i+1}^- \mapsto \theta_i^+, \qquad \theta_i^+ \mapsto \theta_{i+1}^-$$

The weights must transform as

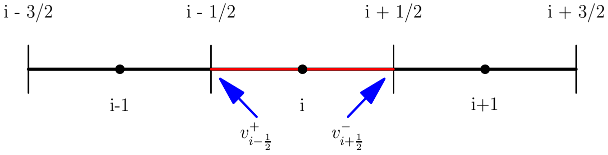
$$w_0 \mapsto \widetilde{w}_1, \qquad w_1 \mapsto \widetilde{w}_0, \qquad \widetilde{w}_0 \mapsto w_1, \qquad \widetilde{w}_1 \mapsto w_0$$

Assuming negation symmetry holds,

$$\text{mirror sym.} \quad \Longleftrightarrow \quad C_1(a,b) = C_2(b,a)$$

Essential properties

Inner jump condition: For each cell i:



$$\text{sign}(v^-_{i+\frac{1}{2}} - v^+_{i-\frac{1}{2}}) = \underbrace{\text{sign}(\Delta v_{i+\frac{1}{2}}) = \text{sign}(\Delta v_{i-\frac{1}{2}})}_{\text{if true}}$$

Automatically holds for consistent WENO-3 weights

Remark: True for ENO-2, but not for higher order ENO.

Explicit weights: SP-WENO

Define

$$\psi^+ := \frac{(1-\theta^-_{i+1})}{(1-\theta^+_i)}, \qquad \psi^- := \frac{1}{\psi^+}$$

Based on the values of θ^-_{i+1} and θ^+_i, choose

$$\begin{aligned} C_1(\theta^+_i, \theta^-_{i+1}) &= \begin{cases} \frac{1}{8}\left(\frac{f^+}{(f^+)^2+(f^-)^2}\right) & \text{if } \theta^+_i \neq 1,\ \psi^+ < 0,\ \psi^+ \neq -1 \\ 0 & \text{if } \theta^+_i \neq 1, \psi^+ = -1 \\ -\frac{3}{8} & \text{if } \theta^+_i = 1 \text{ or } \psi^+ \geq 0,\ |\theta^+_i| \leq 1 \\ \frac{1}{8} & \text{if } \psi^+ \geq 0,\ |\theta^+_i| > 1 \end{cases} \\ C_2(\theta^+_i, \theta^-_{i+1}) &= C_1(\theta^-_{i+1}, \theta^+_i) \end{aligned}$$

where

$$\begin{aligned} f^+(\theta^+_i, \theta^-_{i+1}) &:= \begin{cases} \frac{1}{1+\psi^+} & \text{if } \theta^+_i \neq 1, \psi^+ \neq -1 \\ 1 & \text{otherwise,} \end{cases} \\ f^-(\theta^+_i, \theta^-_{i+1}) &:= f^+(\theta^-_{i+1}, \theta^+_i) \end{aligned}$$

Explicit weights: SP-WENO

The reconstructed jump has the following (simple) expression:

$$
[\![v]\!]_{i+\frac{1}{2}} = \begin{cases}
\frac{1}{2}(\Delta v_{i+\frac{1}{2}} - \Delta v_{i-\frac{1}{2}}) & \text{if} \quad \begin{array}{l} |\theta_i^+| \leq 1 \text{ and } \theta_{i+1}^- = 1 \\ -1 \leq \theta_i^+ < 1 \text{ and } \theta_{i+1}^- < -1 \end{array} \\[2ex]
\frac{1}{2}(\Delta v_{i+\frac{1}{2}} - \Delta v_{i+\frac{3}{2}}) & \text{if} \quad \begin{array}{l} \theta_i^+ = 1 \text{ and } |\theta_{i+1}^-| \leq 1 \\ \theta_i^+ < -1 \text{ and } -1 \leq \theta_{i+1}^- < 1 \end{array} \\[2ex]
\Delta v_{i+\frac{1}{2}} - \frac{1}{2}(\Delta v_{i-\frac{1}{2}} + \Delta v_{i+\frac{3}{2}}) & \text{if} \quad -1 \leq \theta_i^+, \theta_{i+1}^- < 1 \\[2ex]
0 & \text{if} \quad \begin{array}{l} \theta_i^+ > 1 \text{ and } \theta_{i+1}^- > 1 \\ \theta_i^+ < 1 \text{ and } \theta_{i+1}^- > 1 \\ \theta_i^+ > 1 \text{ and } \theta_{i+1}^- < 1 \\ |\theta_i^+| > 1 \text{ and } \theta_{i+1}^- = 1 \\ \theta_i^+ = 1 \text{ and } |\theta_{i+1}^-| > 1 \\ \theta_i^+ < -1 \text{ and } \theta_{i+1}^- < -1 \end{array}
\end{cases}
$$

Explicit weights: SP-WENO

The reconstructed jump has the following (simple) expression:

$$
[\![v]\!]_{i+\frac{1}{2}} = \begin{cases}
\frac{1}{2}(\Delta v_{i+\frac{1}{2}} - \Delta v_{i-\frac{1}{2}}) & \text{if} \quad \begin{array}{l} |\theta_i^+| \le 1 \text{ and } \theta_{i+1}^- = 1 \\ -1 \le \theta_i^+ < 1 \text{ and } \theta_{i+1}^- < -1 \end{array} \\
\frac{1}{2}(\Delta v_{i+\frac{1}{2}} - \Delta v_{i+\frac{3}{2}}) & \text{if} \quad \begin{array}{l} \theta_i^+ = 1 \text{ and } |\theta_{i+1}^-| \le 1 \\ \theta_i^+ < -1 \text{ and } -1 \le \theta_{i+1}^- < 1 \end{array} \\
\Delta v_{i+\frac{1}{2}} - \frac{1}{2}(\Delta v_{i-\frac{1}{2}} + \Delta v_{i+\frac{3}{2}}) & \text{if} \quad -1 \le \theta_i^+, \theta_{i+1}^- < 1 \\
0 & \text{if} \quad \begin{array}{l} \theta_i^+ > 1 \text{ and } \theta_{i+1}^- > 1 \\ \theta_i^+ < 1 \text{ and } \theta_{i+1}^- > 1 \\ \theta_i^+ > 1 \text{ and } \theta_{i+1}^- < 1 \\ |\theta_i^+| > 1 \text{ and } \theta_{i+1}^- = 1 \\ \theta_i^+ = 1 \text{ and } |\theta_{i+1}^-| > 1 \\ \theta_i^+ < -1 \text{ and } \theta_{i+1}^- < -1 \end{array}
\end{cases}
$$

Explicit weights: SP-WENO

Stability estimate:

$$\left| [\![v]\!]_{i+\frac{1}{2}} \right| \leq 2 \left| \Delta v_{i+\frac{1}{2}} \right| \quad \forall \; i \in \mathbb{Z}$$

Explicit weights: SP-WENO

Stability estimate:

$$\left| [\![v]\!]_{i+\frac{1}{2}} \right| \leq 2 \left| \Delta v_{i+\frac{1}{2}} \right| \quad \forall\, i \in \mathbb{Z}$$

For ENO-2, the bounding constant $= 2$
For ENO-3, the bounding constant $= 3.5$

Explicit weights: SP-WENO

Stability estimate:

$$\left| [\![v]\!]_{i+\frac{1}{2}} \right| \leq 2 \left| \Delta v_{i+\frac{1}{2}} \right| \quad \forall\, i \in \mathbb{Z}$$

For ENO-2, the bounding constant $= 2$
For ENO-3, the bounding constant $= 3.5$

Better stability bounds for equivalent accuracy!

Testing reconstruction accuracy

Consider

$$u(x) = \sin(10\pi x) + x, \quad x \in [0,1].$$

Error in the interface values

$$\|u^-_{i+\frac{1}{2}} - u(x_{i+\frac{1}{2}})\|_{L^p_h} + \|u^+_{i+\frac{1}{2}} - u(x_{i+\frac{1}{2}})\|_{L^p_h}, \quad p \in [1,\infty]$$

where

$$\|(.)_i\|_{L^p_h} = \left(\sum_{i=1}^{N} |(.)_i|^p h\right)^{\frac{1}{p}} \quad \text{for } p < \infty, \qquad \|(.)_i\|_{L^\infty_h} = \max_i |(.)_i|$$

Testing reconstruction accuracy

N	SP-WENO L_h^1 error	SP-WENO L_h^1 rate	ENO3 L_h^1 error	ENO3 L_h^1 rate
40	8.59e-02	-	3.95e-02	-
80	6.73e-03	3.67	4.90e-03	3.01
160	5.01e-04	3.75	6.08e-04	3.01
320	3.64e-05	3.78	7.57e-05	3.01
640	2.59e-06	3.81	9.47e-06	3.00
1280	1.82e-07	3.83	1.18e-06	3.01
2560	1.26e-08	3.85	1.47e-07	3.00

N	WENO3 L_h^1 error	WENO3 L_h^1 rate	ENO2 L_h^1 error	ENO2 L_h^1 rate
40	2.04e-01	-	2.35e-01	-
80	4.03e-02	2.34	5.39e-02	2.12
160	7.25e-03	2.48	1.29e-02	2.07
320	1.18e-03	2.62	3.14e-03	2.03
640	1.77e-04	2.74	7.76e-04	2.02
1280	2.13e-05	3.05	1.93e-04	2.01
2560	2.10e-06	3.34	4.81e-05	2.00

Evolution problems

Numerical flux:

- **EC:** Fourth order entropy conservative flux
- **ES:** SP-WENO, ENO-2 or ENO-3 (all preserve sign)

Time integration using a Strong Stability Preserving RK3 scheme

A sign preserving WENO reconstruction method; U. S. Fjordholm, D. Ray ; Journal of Scientific Computing, Vol. 68(1), pp. 42-63 (2015)

(scalar conservation laws)

Burgers' equations

$$u_t + uu_x = 0, \qquad d_{i+\frac{1}{2}} = \frac{|u_i| + |u_{i+1}|}{2}$$

Test: Smooth solution

$$\Omega = [-1, 1], \quad T_f = 0.3, \quad CFL = 0.4, \quad u_0(x) = 1 + \frac{1}{2}\sin(\pi x)$$

N	SP-WENO L^1_h error	SP-WENO L^1_h rate	ENO3 L^1_h error	ENO3 L^1_h rate	ENO2 L^1_h error	ENO2 L^1_h rate
50	3.41e-04	-	3.07e-04	-	4.73e-03	-
100	4.17e-05	3.03	4.76e-05	2.69	1.35e-03	1.81
200	4.51e-06	3.21	8.44e-06	2.49	3.77e-04	1.84
400	4.98e-07	3.18	1.80e-06	2.23	1.02e-04	1.89
600	1.33e-07	3.26	7.29e-07	2.23	4.71e-05	1.90
800	5.22e-08	3.25	3.91e-07	2.17	2.72e-05	1.92

Euler equations

$$\mathbf{U} = \begin{bmatrix} \rho \\ \rho u \\ E \end{bmatrix}, \quad \mathbf{f}(\mathbf{U}) = \begin{bmatrix} \rho u \\ \rho u^2 + p \\ u(E+p) \end{bmatrix}$$

where

$$E = \rho\left(\frac{1}{2}u^2 + \frac{p}{(\gamma-1)\rho}\right)$$

$$\mathbf{U} = \begin{bmatrix} \rho \\ \rho u \\ E \end{bmatrix}, \quad \mathbf{f}(\mathbf{U}) = \begin{bmatrix} \rho u \\ \rho u^2 + p \\ u(E+p) \end{bmatrix}$$

where

$$E = \rho \left(\frac{1}{2} u^2 + \frac{p}{(\gamma - 1)\rho} \right)$$

Choose

$$\eta(\mathbf{U}) = -\frac{\rho s}{\gamma - 1}, \quad q(\mathbf{U}) = -\frac{\rho u s}{\gamma - 1} \quad \text{(Harten)}$$

$$s = \ln(p) - \gamma \ln(\rho) \quad \text{(physical entropy)}$$

$$\mathbf{V} = \begin{bmatrix} \frac{\gamma - s}{\gamma - 1} - \beta u^2 \\ 2\beta u \\ -2\beta \end{bmatrix}, \qquad \beta = \frac{\rho}{2p} = \frac{1}{RT}$$

Euler equations

EC : Kinetic Energy and Entropy Conservative flux (Chandrashekar)
ES : Roe type dissipation matrix

Test: 1D Sine-wave on $[0,2]$

$$\begin{aligned} \rho &= 1 + 0.5\sin(\pi(x)) \\ u &= 0.5 \\ p &= 1 \end{aligned}$$

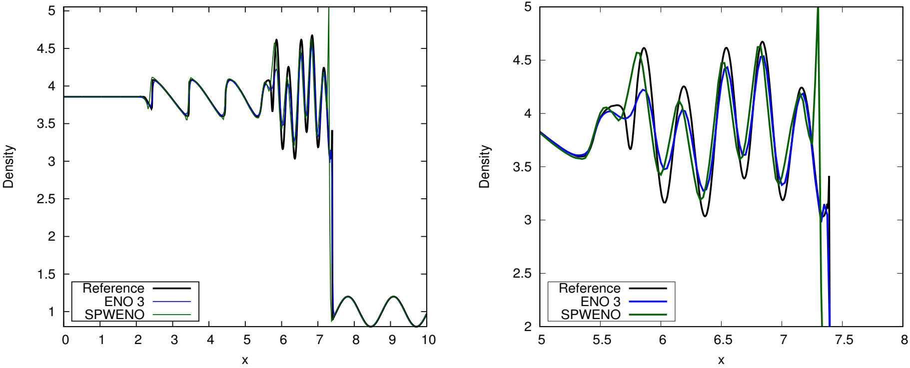
	SP-WENO		ENO-3	
N	L_h^1		L_h^1	
	error	rate	error	rate
50	2.09e-04	-	2.93e-04	-
100	2.57e-05	3.06	3.70e-05	2.98
200	2.86e-06	3.13	4.65e-06	2.99
400	3.19e-07	3.16	5.81e-07	2.99
800	3.49e-08	3.19	7.27e-8	2.99

Euler equations

Test: Shu-Osher test case on $[-5, 5]$ with $N = 400$

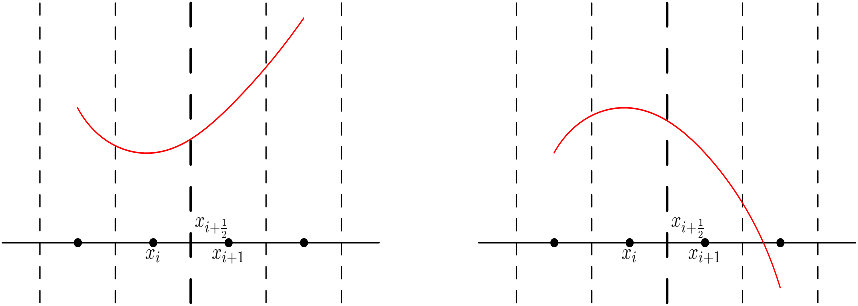
$$\rho = \begin{cases} 3.857143 & \text{if } x < -4 \\ 1.0 + 0.2\sin(5x) & \text{if } x > -4. \end{cases}, \quad u = \begin{cases} 2.629369 & \text{if } x < -4 \\ 0 & \text{if } x > -4. \end{cases}$$

$$p = \begin{cases} 10.33333 & \text{if } x < -4 \\ 1.0 & \text{if } x > -4. \end{cases}$$



Problem with SP-WENO

$$[\![v]\!]_{i+\frac{1}{2}} = \frac{1}{2}\left[\widetilde{w}_0(1-\theta^-_{i+1}) + w_1(1-\theta^+_i)\right]\Delta v_{i+\frac{1}{2}}$$



$$\left[\widetilde{w}_0(1-\theta^-_{i+1}) + w_1(1-\theta^+_i)\right] = 0$$

A fix to SP-WENO

$$[\![v]\!]_{i+\frac{1}{2}} = \frac{1}{2}\left[\widetilde{w}_0(1-\theta^-_{i+1}) + w_1(1-\theta^+_i) + \left(\frac{|\Delta v_{i+\frac{1}{2}}|}{0.5(|v_i|+|v_{i+1}|)}\right)^3\right]\Delta v_{i+\frac{1}{2}}$$

Pulling correction into weights

$$C_1 \to C_1 - \frac{1}{4}\left(\frac{|\Delta v_{i+\frac{1}{2}}|}{0.5(|v_i|+|v_{i+1}|)}\right)^3 \frac{1}{(1-\theta^+_i)}$$

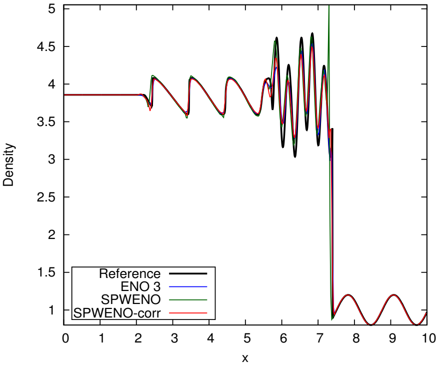
$$C_2 \to C_2 - \frac{1}{4}\left(\frac{|\Delta v_{i+\frac{1}{2}}|}{0.5(|v_i|+|v_{i+1}|)}\right)^3 \frac{1}{(1-\theta^-_{i+1})}$$

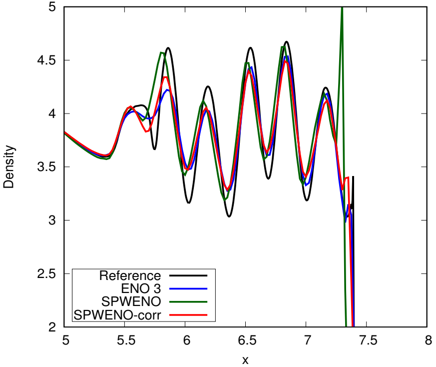
To ensure consistency

$$C_i = \min(\max(C_i, -3/8), 1/8)$$

A fix to SP-WENO

Test: Shu-Osher





A fix to SP-WENO

Test: 1D Sine-wave

N	SP-WENO		SP-WENO-corr	
	L^1_h		L^1_h	
	error	rate	error	rate
50	2.09e-04	-	2.05e-04	-
100	2.57e-05	3.06	2.42e-05	3.09
200	2.86e-06	3.13	2.79e-6	3.10
400	3.19e-07	3.16	3.15e-7	3.15
800	3.49-08	3.19	3.47e-8	3.18

A fix to SP-WENO

Test: 2D Sine-wave on on $[0,2] \times [0,2]$

$$\begin{aligned} \rho &= 1 + 0.5\sin(\pi(x+y)) \\ u &= 0.5 \\ v &= 0.8 \\ p &= 1 \end{aligned}$$

$N_x \times N_y$	SP-WENO-corr L_h^1 error	SP-WENO-corr L_h^1 rate
100 × 100	7.60e-04	-
200 × 200	8.38e-05	3.18
300 × 300	2.35e-05	3.14
400 × 400	9.51e-06	3.14
500 × 500	4.67e-07	3.18

A fix to SP-WENO

Properties retained:

- ✓ Consistency
- ✓ Sign-property
- ✓ Negation symmetry
- ✓ Mirror property
- ✓ Inner jump condition
- × Bound on reconstructed jump

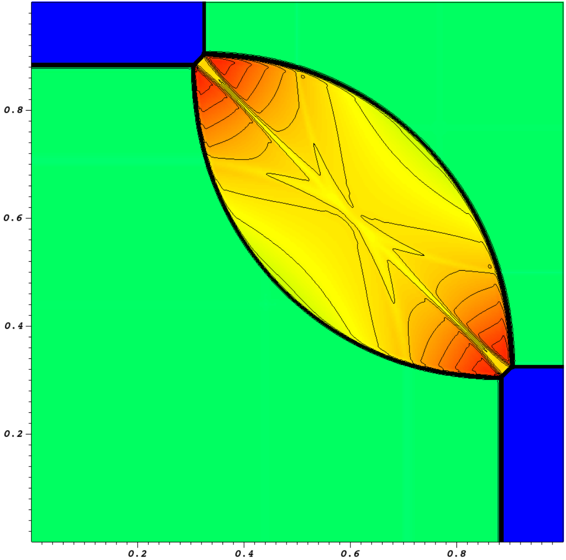
SP-WENO-corr

Test: 2D Riemann problem (4 shocks)
Domain: $[0,1] \times [0,1]$ with 400×400 cell
Initial condition:

$$\begin{aligned} \rho_2 &= 0.5065 \\ u_2 &= 0.8938 \\ v_2 &= 0 \\ p_2 &= 0.35 \end{aligned} \qquad \begin{aligned} \rho_1 &= 1.1 \\ u_1 &= 0 \\ v_1 &= 0 \\ p_1 &= 1.1 \end{aligned}$$

$$\begin{aligned} \rho_3 &= 1.1 \\ u_3 &= 0.8938 \\ v_3 &= 0.8938 \\ p_3 &= 1.1 \end{aligned} \qquad \begin{aligned} \rho_4 &= 0.5065 \\ u_4 &= 0 \\ v_4 &= 0.8939 \\ p_4 &= 0.35 \end{aligned}$$

Test: 2D Riemann problem (4 shocks)



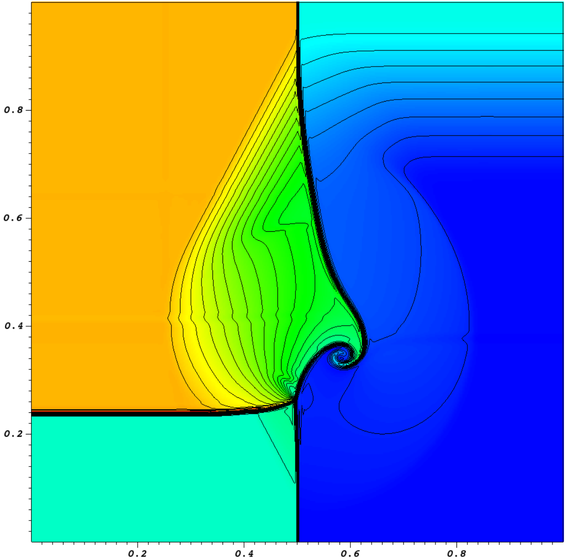
Test: 2D Riemann problem (shock + rarefaction + 2 contacts)
Domain: $[0,1] \times [0,1]$ with 400×400 cell
Initial condition:

$$\begin{aligned} \rho_2 &= 2 \\ u_2 &= 0 \\ v_2 &= -0.3 \\ p_2 &= 1 \end{aligned} \qquad \begin{aligned} \rho_1 &= 1.0 \\ u_1 &= 0 \\ v_1 &= 0.3 \\ p_1 &= 1.0 \end{aligned}$$

$$\begin{aligned} \rho_3 &= 1.0625 \\ u_3 &= 0 \\ v_3 &= 0.2145 \\ p_3 &= 0.4 \end{aligned} \qquad \begin{aligned} \rho_4 &= 0.5197 \\ u_4 &= 0 \\ v_4 &= -0.4259 \\ p_4 &= 0.4 \end{aligned}$$

SP-WENO-corr

Test: 2D Riemann problem (shock + rarefaction + 2 contacts)



Conclusion

Proposed a WENO method satisfying

- Sign property (needed for entropy stability)
- Symmetries (mirror, negation)
- Third (or higher) order accurate
- Added correction for extension to systems

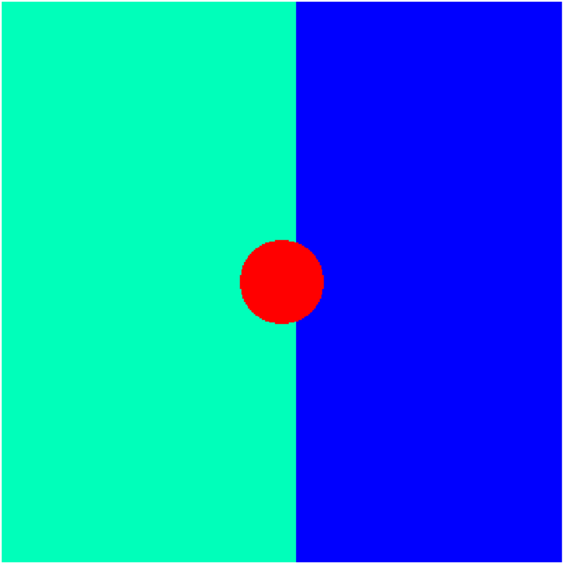
Conclusion

Proposed a WENO method satisfying

- Sign property (needed for entropy stability)
- Symmetries (mirror, negation)
- Third (or higher) order accurate
- Added correction for extension to systems

What's next?

- Modifying correction to recover bounds?
- Higher order WENO with similar properties?



Questions?

Constructing an entropy stable scheme

Choosing

$$\eta(u) = \frac{u^2}{2} \quad \Longrightarrow \quad v(u) = u$$

Examples:

- Linear advection: $f(u) = cu$

$$F^*_{i+\frac{1}{2}} = c\frac{u_i + u_{i+1}}{2}$$

- Burgers' equation: $f(u) = u^2/2$

$$F^*_{i+\frac{1}{2}} = \frac{u_i^2 + u_{i+1}^2 + u_i u_{i+1}}{6}$$

Entropy conservative flux for Euler

Kinetic energy and entropy conservative flux

$$\mathbf{F}^* = \begin{bmatrix} F^{*,\rho} \\ F^{*,m_1} \\ F^{*,m_2} \\ F^{*,e} \end{bmatrix} = \begin{bmatrix} \widehat{\rho}\overline{u}_n \\ \tilde{p}n_1 + \overline{u}F^{*,\rho} \\ \tilde{p}n_2 + \overline{v}F^{*,\rho} \\ F^{*,e} \end{bmatrix}$$

where

$$\overline{u}_n = \overline{u}n_1 + \overline{v}n_2, \qquad \tilde{p} = \frac{\overline{\rho}}{2\overline{\beta}},$$

$$F^{*,e} = \left[\frac{1}{2(\gamma-1)\widehat{\beta}} - \frac{1}{2}\overline{|\mathbf{u}|^2}\right] F^{*,\rho} + \overline{\mathbf{u}}\cdot \mathbf{F}^{*,m}$$

The crucial property for kinetic energy preservation (Jameson)

$$\mathbf{F}^m = p\mathbf{n} + \overline{\mathbf{u}}F^\rho$$

for any consistent approximations of p and F^ρ.

Kinetic energy preservation

$$K = \frac{1}{2}\rho|\mathbf{u}|^2$$

$$\begin{aligned}
\frac{\partial K}{\partial t} &= -\frac{1}{2}|\mathbf{u}|^2\frac{\partial \rho}{\partial t} + \left\langle \mathbf{u}, \frac{\partial(\rho\mathbf{u})}{\partial t}\right\rangle \\
&= -\frac{1}{2}\frac{\partial\left(\rho u_1^3 + \rho u_1 u_2^2 + 2u_1 p\right)}{\partial x} - \frac{1}{2}\frac{\partial\left(\rho u_2^3 + \rho u_1^2 u_2 + 2u_2 p\right)}{\partial y} + p\nabla\cdot\mathbf{u}
\end{aligned}$$

Global kinetic energy evolves by

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega K\,\mathrm{d}\mathbf{x} = \int_\Omega p\nabla\cdot\mathbf{u}\,\mathrm{d}\mathbf{x}$$

Kinetic energy preservation $\longrightarrow$ scheme satisfies discrete analogue

Entropy stable KEPES flux

For Euler equations, we choose **Roe type dissipation**

$$\mathbf{D}_{i+\frac{1}{2}} = \mathbf{R}_{i+\frac{1}{2}} |\Lambda_{i+\frac{1}{2}}| \mathbf{R}_{i+\frac{1}{2}}^\top$$

where

$$\mathbf{R} = \begin{bmatrix} 1 & 1 & 0 & 1 \\ u - a\tilde{n}_1 & u & \tilde{n}_2 & u + a\tilde{n}_1 \\ v - a\tilde{n}_2 & v & -\tilde{n}_1 & v + a\tilde{n}_2 \\ H - au_{\tilde{n}} & \frac{1}{2}|\mathbf{u}|^2 & u\tilde{n}_2 - v\tilde{n}_1 & H + au_{\tilde{n}} \end{bmatrix} \mathbf{S}^{\frac{1}{2}}$$

$$\mathbf{S} = \text{diag}\left[\frac{\rho}{2\gamma}, \quad \frac{(\gamma-1)\rho}{\gamma}, \quad p, \quad \frac{\rho}{2\gamma}\right]$$

$$\Lambda = \text{diag}\left[u_n - a, \quad u_n, \quad u_n, \quad u_n + a\right]$$

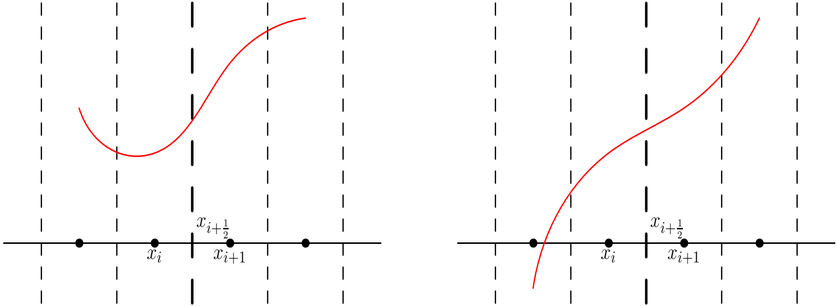
$\tilde{\mathbf{n}} \rightarrow$ unit face normal, $\quad u_{\tilde{n}} = \mathbf{u} \cdot \tilde{\mathbf{n}}$

$a = \sqrt{\frac{\gamma p}{\rho}} \rightarrow$ speed of sound in air

$H = \frac{a^2}{\gamma - 1} + \frac{|\mathbf{u}|^2}{2} \rightarrow$ specific enthalpy

Essential properties

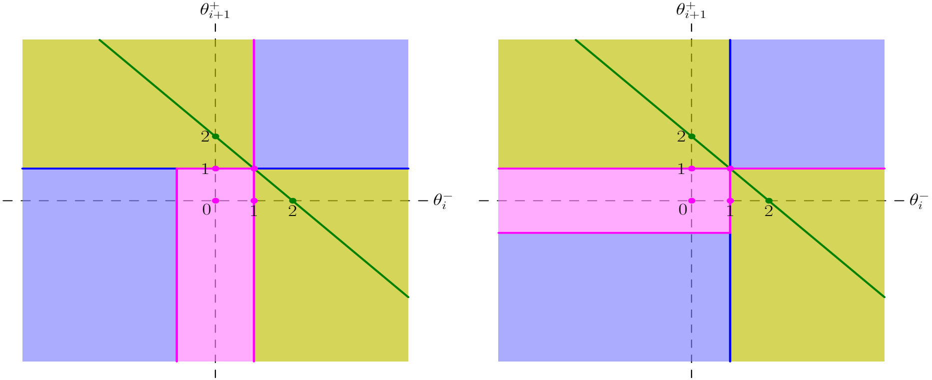
Variable order: If $v''(x) = 0$ for $|x - x_{i+\frac{1}{2}}| = \mathcal{O}(h)$, then ENO-2 polynomials are third order accurate.



$$C_1, C_2 = \begin{cases} \mathcal{O}(h), & \text{in GC} \\ \text{no restriction}, & \text{in SC} \end{cases}$$

SP-WENO weights

LABEL	Left fig: w_1	Right fig: $\widetilde{w}_0$
blue	0	0
magenta	1	1
green	$\frac{1}{4}$	$\frac{1}{4}$
yellow	$\frac{1}{4}\left(1 - \frac{f^+}{(f^+)^2+(f^-)^2}\right)$	$\frac{1}{4}\left(1 - \frac{f^-}{(f^+)^2+(f^-)^2}\right)$



A second fix to SP-WENO

Testing reconstruction accuracy

$$u(x) = \sin(10\pi x) + x, \quad x \in [0,1].$$

N	SP-WENO		SP-WENO-corr	
	L^1_h		L^1_h	
	error	rate	error	rate
40	8.59e-02	-	1.44e-01	-
80	6.73e-03	3.67	2.19e-02	2.71
160	5.01e-04	3.75	3.47e-03	2.66
320	3.64e-05	3.78	5.43e-04	2.67
640	2.59e-06	3.81	8.49e-05	2.67
1280	1.82e-07	3.83	1.32e-05	2.68
2560	1.26e-08	3.85	2.08e-06	2.67

A second fix to SP-WENO

Translating away from 0

$$u(x) = 1 + \sin(10\pi x) + x, \quad x \in [0, 1].$$

N	SP-WENO		SP-WENO-corr	
	L_h^1		L_h^1	
	error	rate	error	rate
40	8.58e-02	-	9.81e-02	-
80	6.73e-03	3.67	8.25e-03	3.56
160	5.01e-04	3.75	6.17e-04	3.74
320	3.64e-05	3.78	4.38e-05	3.81
640	2.59e-06	3.81	3.06e-06	3.84
1280	1.82e-07	3.83	2.11e-07	3.86
2560	1.26e-08	3.85	1.44e-08	3.87

A second fix to SP-WENO

$$[\![v]\!]_{i+\frac{1}{2}} = \frac{1}{2}\left[\widetilde{w}_0(1-\theta^-_{i+1}) + w_1(1-\theta^+_i) + \left(\frac{|\Delta v_{i+\frac{1}{2}}|}{0.5(|v_i|+|v_{i+1}|)}\right)^3\right]\Delta v_{i+\frac{1}{2}}$$

Denominator can be ≈ 0

Bound the correction

$$\left(\frac{|\Delta v_{i+\frac{1}{2}}|}{0.5(|v_i|+|v_{i+1}|)}\right)^3 \quad \longrightarrow \quad \min\left(\frac{|\Delta v_{i+\frac{1}{2}}|}{0.5(|v_i|+|v_{i+1}|)}, |\Delta v_{i+\frac{1}{2}}|\right)^3$$

A second fix to SP-WENO

$$u(x) = \sin(10\pi x) + x, \quad x \in [0,1].$$

N	SP-WENO		SP-WENO-corr		SP-WENO-corr2	
	L_h^1		L_h^1		L_h^1	
	error	rate	error	rate	error	rate
40	8.59e-02	-	1.44e-01	-	8.67e-02	-
80	6.73e-03	3.67	2.19e-02	2.71	6.88e-03	3.65
160	5.01e-04	3.75	3.47e-03	2.66	5.09e-04	3.76
320	3.64e-05	3.78	5.43e-04	2.67	3.68e-05	3.79
640	2.59e-06	3.81	8.49e-05	2.67	2.61e-06	3.81
1280	1.82e-07	3.83	1.32e-05	2.68	1.83e-07	3.83
2560	1.26e-08	3.85	2.08e-06	2.67	1.27e-08	3.85

A second fix to SP-WENO

Test: Shu-Osher