A sign preserving WENO reconstruction

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ICIAM-2015, Beijing
14th August, 2015

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Supported by:

- AIRBUS Group Corporate Foundation Chair in Mathematics of Complex Systems, established in TIFR/ICTS, Bangalore

Conservation laws and entropy conditions

Consider the 1D Cauchy problem

$$\begin{aligned} \partial_t u + \partial_x f(u) &= 0 \qquad && \forall \quad (x,t) \in \mathbb{R} \times \mathbb{R}^+ \\ u(x,0) &= u_0(x) \qquad && \forall \quad x \in \mathbb{R} \end{aligned}$$

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Search for Weak (distributional) Solution.

Uniqueness?

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Entropy-entropy flux pair $(\eta(u), q(u))$ with $q' = \eta' f'$.

$$v(u) = \eta'(u) \rightarrow \text{entropy variables}$$

$$v(u)\partial_t u + v(u)\partial_x(f(u)) = \partial_t \eta(u) + \partial_x q(u) = 0$$

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For discontinuous solutions we have

$$\boxed{\partial_t \eta(u) + \partial_x q(u) \leq 0}$$

Existence, uniqueness of solutions for scalar conservation laws (Kruzkov).

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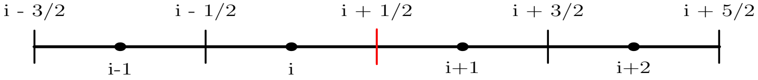
Outline

- Finite difference scheme
- Entropy conservative/stable schemes
- Higher-order entropy stable schemes
- Sign preservation and other essentials
- SP-WENO
- Numerical results

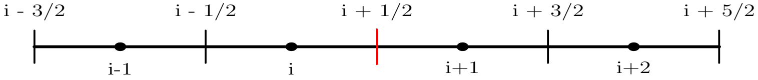
Finite difference scheme



Discretize the domain $\Omega = \bigcup_i I_i$, where

$$I_i = [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}), \qquad x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}} \equiv h$$

Finite difference scheme



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Consider the semi-discrete finite difference scheme

$$\frac{\mathrm{d}u_i}{\mathrm{d}t} + \frac{1}{h}\left(F_{i+\frac{1}{2}} - F_{i-\frac{1}{2}}\right) = 0$$

$F_{i+\frac{1}{2}}(t) = F(u_i(t), u_{i+1}(t))$ is the numerical flux satisfying

1. Consistency: $F(u,u) = f(u)$
2. Conservation: $F(u_1,u_2) = -F(u_2,u_1)$

Constructing an entropy stable scheme

Step 1: Entropy conservation

Entropy conservative scheme

A scheme is entropy conservative if

$$\frac{\mathrm{d}\eta(u_i)}{\mathrm{d}t} + \frac{1}{h}\left(q^*_{i+\frac{1}{2}} - q^*_{i-\frac{1}{2}}\right) = 0$$

where $q^*_{i+\frac{1}{2}}$ is a consistent numerical entropy flux.

Constructing an entropy stable scheme

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Notation: $\Delta(\cdot)_{i+\frac{1}{2}} = (\cdot)_{i+1} - (\cdot)_i$

A sufficient condition (Tadmor)

A scheme with flux $F^*_{i+\frac{1}{2}}$ is entropy conservative if

$$\Delta v_{i+\frac{1}{2}} F^*_{i+\frac{1}{2}} = \Delta \Psi_{i+\frac{1}{2}}$$

where $\Psi(u) := v(u) f(u) - q(u)$ is the entropy potential.

Constructing an entropy stable scheme

Remarks

- The above scheme is second order accurate in space.
- Unique entropy conservative flux (given η)

$$F^*_{i+\frac{1}{2}} = \frac{\Delta \Psi_{i+\frac{1}{2}}}{\Delta v_{i+\frac{1}{2}}}.$$

- Higher order entropy conservative schemes (Mercier et al.)

$$F^{*,2p}_{i+\frac{1}{2}} = \sum_{r=1}^{p} \alpha_r^p \sum_{s=0}^{r-1} F^*(u_{i-s}, u_{i-s+r}) \qquad (2p\text{-th order})$$

- For scalar laws, any convex $\eta(u)$ works, with

$$q(u) = \int^{u} \eta'(z) f'(z) dz$$

Constructing an entropy stable scheme

Step 2: Add dissipation
Entropy dissipated near discontinuities.

$$F_{i+\frac{1}{2}} = F^{*,2p}_{i+\frac{1}{2}} - \frac{1}{2} a_{i+\frac{1}{2}} \Delta v_{i+\frac{1}{2}}$$

where $a_{i+\frac{1}{2}} \approx |f'(u)|$.

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Entropy stable scheme

The scheme with the above flux satisfies

$$\frac{\mathrm{d}\eta(u_i)}{\mathrm{d}t} + \frac{1}{h}\left(q_{i+\frac{1}{2}} - q_{i-\frac{1}{2}}\right) \leq 0$$

High-order diffusion

$$\Delta v_{i+\frac{1}{2}} \sim \mathcal{O}(h) \quad \Longrightarrow \quad \text{first order flux}$$

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Idea: At each $x_{i+\frac{1}{2}}$, reconstruct v in I_i, I_{i+1} with polynomials $v_i(x), v_{i+1}(x)$ respectively.

$$v^-_{i+\frac{1}{2}} = v_i(x_{i+\frac{1}{2}}), \qquad v^+_{i+\frac{1}{2}} = v_{i+1}(x_{i+\frac{1}{2}}), \qquad [\![v]\!]_{i+\frac{1}{2}} = v^+_{i+\frac{1}{2}} - v^-_{i+\frac{1}{2}}$$

Replace $\Delta v_{i+\frac{1}{2}}$ by $[\![v]\!]_{i+\frac{1}{2}} \sim \mathcal{O}(h^k)$.

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How do we ensure entropy stability?

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Replace $\Delta v_{i+\frac{1}{2}}$ by $[\![v]\!]_{i+\frac{1}{2}} \sim \mathcal{O}(h^k)$.

Sign property (Fjordholm et al.)

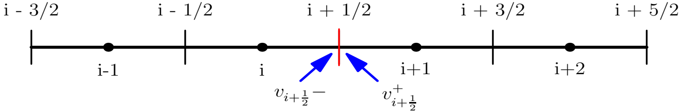
The scheme with the numerical flux

$$F_{i+\frac{1}{2}} = F^*_{i+\frac{1}{2}} - \frac{1}{2} a_{i+\frac{1}{2}} [\![v]\!]_{i+\frac{1}{2}}$$

is entropy stable if the following sign property holds for $x_{i+\frac{1}{2}}$

$$\boxed{\mathrm{sign}([\![v]\!]_{i+\frac{1}{2}}) = \mathrm{sign}(\Delta v_{i+\frac{1}{2}}).}$$

Second order reconstruction with minmod limiter



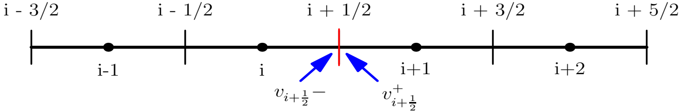
$$
\begin{aligned}
v^{-}_{i+\frac{1}{2}} &= v_i + \frac{1}{2}\mathsf{minmod}(\Delta v_{i+\frac{1}{2}}, \Delta v_{i-\frac{1}{2}}) \\
v^{+}_{i+\frac{1}{2}} &= v_{i+1} - \frac{1}{2}\mathsf{minmod}(\Delta v_{i+\frac{3}{2}}, \Delta v_{i+\frac{1}{2}})
\end{aligned}
$$

where

$$
\mathsf{minmod}(a,b) = \begin{cases} \mathrm{sign}(a)\min\left(|a|,|b|\right), & \text{if } \mathrm{sign}(a) = \mathrm{sign}(b) \\ 0, & \text{o.w.} \end{cases}
$$

Sign preserving reconstructions

Second order reconstruction with minmod limiter



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ENO interpolation (Fjordholm et. al)

Construct k-th degree polynomial using adaptive stencil.

WENO reconstruction

Idea: Convex combination of all k polynomials of (ENO-k) to get a (2k-1)-th order approximation.

Advantages:

- Full utilization of (2k-1) cells
- ENO has accuracy issues due to unstable stencils [1]

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Choose weights to satisfy the sign property?

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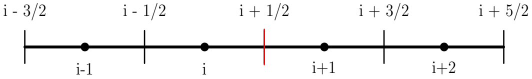
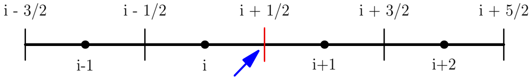
WENO-3





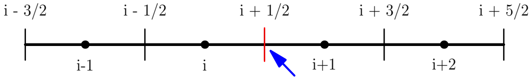
Reconstruction from left: $v^-_{i+\frac{1}{2}}$

$$S_0 = \{x_i, x_{i+1}\}, \quad S_1 = \{x_{i-1}, x_i\}$$

Second order reconstructions are:

$$v^{(0),-}_{i+\frac{1}{2}} = \frac{v_i}{2} + \frac{v_{i+1}}{2}, \qquad v^{(1),-}_{i+\frac{1}{2}} = -\frac{v_{i-1}}{2} + \frac{3v_i}{2}$$

WENO-3



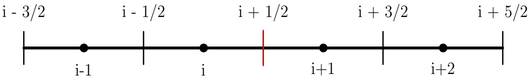
Reconstruction from right: $v^{+}_{i+\frac{1}{2}}$

$$S_0 = \{x_{i+1}, x_{i+2}\}, \quad S_1 = \{x_i, x_{i+1}\}$$

Second order reconstructions are:

$$v^{(0),+}_{i+\frac{1}{2}} = \frac{3v_{i+1}}{2} - \frac{v_{i+2}}{2}, \qquad v^{(1),+}_{i+\frac{1}{2}} = \frac{v_i}{2} + \frac{v_{i+1}}{2}$$

WENO-3



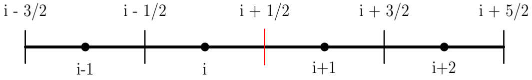
WENO-3: convex combination of ENO-2 polynomials

$$v^{-}_{i+\frac{1}{2}} = w_0 v^{(0),-}_{i+\frac{1}{2}} + w_1 v^{(1),-}_{i+\frac{1}{2}}, \qquad v^{+}_{i+\frac{1}{2}} = \tilde{w}_0 v^{(0),+}_{i+\frac{1}{2}} + \tilde{w}_1 v^{(1),+}_{i+\frac{1}{2}}$$

Taylor expansion gives the (order) constraints

$$w_0 + w_1 = 1 \qquad \widetilde{w}_0 + \widetilde{w}_1 = 1$$

$$\frac{w_0}{8} - \frac{3w_1}{8} = C_1 = \mathcal{O}(h) \qquad -\frac{\widetilde{3w_0}}{8} + \frac{\widetilde{w}_1}{8} = C_2 = \mathcal{O}(h)$$



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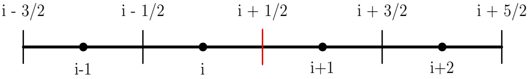
$$\boxed{\begin{array}{ll} w_0 = \dfrac{3}{4} + 2C_1 & \qquad w_1 = \dfrac{1}{4} - 2C_1 \\ \widetilde{w}_0 = \dfrac{1}{4} - 2C_2 & \qquad \widetilde{w}_1 = \dfrac{3}{4} + 2C_2 \end{array}}$$



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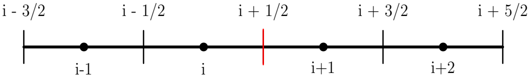
"Ideal" weights

WENO-3



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"Ideal" weights

Issues:

- No smoothness indicators
- Sign property

Essential properties

Consistency:

$$0 \leq w_0, w_1, \widetilde{w}_0, \widetilde{w}_1 \leq 1$$

or

$$-\frac{3}{8} \leq C_1, C_2 \leq \frac{1}{8}$$

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or

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Sign property:

$$\mathsf{sign}(\Delta v_{i+\frac{1}{2}}) = \mathsf{sign}([\![v]\!]_{i+\frac{1}{2}})$$

Define,

$$\theta_i^- = \frac{\Delta v_{i+\frac{1}{2}}}{\Delta v_{i-\frac{1}{2}}}, \qquad \theta_i^+ = \frac{1}{\theta_i^-} = \frac{\Delta v_{i-\frac{1}{2}}}{\Delta v_{i+\frac{1}{2}}}$$

Thus,

$$\left[\widetilde{w}_0(1 - \theta_{i+1}^-) + w_1(1 - \theta_i^+)\right] \geq 0$$

Essential properties

Negation symmetry: No bias towards positive or negative values.
Under the transform $v \mapsto -v$

$$\Delta v_{j+\frac{1}{2}} \mapsto -\Delta v_{j+\frac{1}{2}} \qquad \forall\ j \in \mathbb{Z}$$

But θ_j^- or θ_j^+ remain unchanged.

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But θ_j^- or θ_j^+ remain unchanged.
A sufficient condition

$$C_1 = C_1(\theta_i^+, \theta_{i+1}^-), \qquad C_2 = C_2(\theta_i^+, \theta_{i+1}^-)$$

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Mirror property: Mirroring solution about $x_{i+\frac{1}{2}}$ gives

$$\theta_{i+1}^- \mapsto \theta_i^+, \qquad \theta_i^+ \mapsto \theta_{i+1}^-$$

The weights must transform as

$$w_0 \mapsto \widetilde{w}_1, \qquad w_1 \mapsto \widetilde{w}_0, \qquad \widetilde{w}_0 \mapsto w_1, \qquad \widetilde{w}_1 \mapsto w_0$$

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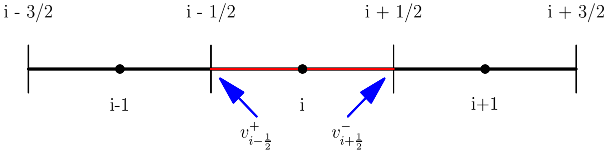
Assuming negation symmetry holds,

$$\text{mirror sym.} \quad \Longleftrightarrow \quad C_1(a,b) = C_2(b,a)$$

Essential properties

Inner jump condition: For each cell i:



$$\text{sign}(v^-_{i+\frac{1}{2}} - v^+_{i-\frac{1}{2}}) = \underbrace{\text{sign}(\Delta v_{i+\frac{1}{2}}) = \text{sign}(\Delta v_{i-\frac{1}{2}})}_{\text{if true}}$$

Automatically holds for consistent WENO-3 weights

Remark: True for ENO-2, but not for higher order ENO.

Explicit weights: SP-WENO

Define

$$\psi^+ := \frac{(1-\theta^-_{i+1})}{(1-\theta^+_i)}, \qquad \psi^- := \frac{1}{\psi^+}$$

Based on the values of θ^-_{i+1} and θ^+_i, choose

$$C_1(\theta^+_i, \theta^-_{i+1}) = \begin{cases} \frac{1}{8}\left(\frac{f^+}{(f^+)^2+(f^-)^2}\right) & \text{if } \theta^+_i \neq 1,\ \psi^+ < 0,\ \psi^+ \neq -1 \\ 0 & \text{if } \theta^+_i \neq 1, \psi^+ = -1 \\ -\frac{3}{8} & \text{if } \theta^+_i = 1 \text{ or } \psi^+ \geq 0,\ |\theta^+_i| \leq 1 \\ \frac{1}{8} & \text{if } \psi^+ \geq 0,\ |\theta^+_i| > 1 \end{cases}$$

$$C_2(\theta^+_i, \theta^-_{i+1}) = C_1(\theta^-_{i+1}, \theta^+_i)$$

where

$$f^+(\theta^+_i, \theta^-_{i+1}) := \begin{cases} \frac{1}{1+\psi^+} & \text{if } \theta^+_i \neq 1, \psi^+ \neq -1 \\ 1 & \text{otherwise,} \end{cases}$$

$$f^-(\theta^+_i, \theta^-_{i+1}) := f^+(\theta^-_{i+1}, \theta^+_i)$$

Explicit weights: SP-WENO

The reconstructed jump has the following (simple) expression:

$$
[\![v]\!]_{i+\frac{1}{2}} = \begin{cases}
0 & \text{if} \quad \begin{array}{l} \theta_i^+ > 1 \text{ and } \theta_{i+1}^- > 1 \\ \theta_i^+ < 1 \text{ and } \theta_{i+1}^- > 1 \\ \theta_i^+ > 1 \text{ and } \theta_{i+1}^- < 1 \\ |\theta_i^+| > 1 \text{ and } \theta_{i+1}^- = 1 \\ \theta_i^+ = 1 \text{ and } |\theta_{i+1}^-| > 1 \\ \theta_i^+ < -1 \text{ and } \theta_{i+1}^- < -1 \end{array} \\
\frac{1}{2}(\Delta v_{i+\frac{1}{2}} - \Delta v_{i-\frac{1}{2}}) & \text{if} \quad \begin{array}{l} |\theta_i^+| \leq 1 \text{ and } \theta_{i+1}^- = 1 \\ -1 \leq \theta_i^+ < 1 \text{ and } \theta_{i+1}^- < -1 \end{array} \\
\frac{1}{2}(\Delta v_{i+\frac{1}{2}} - \Delta v_{i+\frac{3}{2}}) & \text{if} \quad \begin{array}{l} \theta_i^+ = 1 \text{ and } |\theta_{i+1}^-| \leq 1 \\ \theta_i^+ < -1 \text{ and } -1 \leq \theta_{i+1}^- < 1 \end{array} \\
\Delta v_{i+\frac{1}{2}} - \frac{1}{2}(\Delta v_{i-\frac{1}{2}} + \Delta v_{i+\frac{3}{2}}) & \text{if} \quad -1 \leq \theta_i^+, \theta_{i+1}^- < 1
\end{cases}
$$

Explicit weights: SP-WENO

The reconstructed jump has the following (simple) expression:

$$
[\![v]\!]_{i+\frac{1}{2}} = \begin{cases}
0 & \text{if} \quad \begin{array}{l} \theta_i^+ > 1 \text{ and } \theta_{i+1}^- > 1 \\ \theta_i^+ < 1 \text{ and } \theta_{i+1}^- > 1 \\ \theta_i^+ > 1 \text{ and } \theta_{i+1}^- < 1 \\ |\theta_i^+| > 1 \text{ and } \theta_{i+1}^- = 1 \\ \theta_i^+ = 1 \text{ and } |\theta_{i+1}^-| > 1 \\ \theta_i^+ < -1 \text{ and } \theta_{i+1}^- < -1 \end{array} \\
\frac{1}{2}(\Delta v_{i+\frac{1}{2}} - \Delta v_{i-\frac{1}{2}}) & \text{if} \quad \begin{array}{l} |\theta_i^+| \leq 1 \text{ and } \theta_{i+1}^- = 1 \\ -1 \leq \theta_i^+ < 1 \text{ and } \theta_{i+1}^- < -1 \end{array} \\
\frac{1}{2}(\Delta v_{i+\frac{1}{2}} - \Delta v_{i+\frac{3}{2}}) & \text{if} \quad \begin{array}{l} \theta_i^+ = 1 \text{ and } |\theta_{i+1}^-| \leq 1 \\ \theta_i^+ < -1 \text{ and } -1 \leq \theta_{i+1}^- < 1 \end{array} \\
\Delta v_{i+\frac{1}{2}} - \frac{1}{2}(\Delta v_{i-\frac{1}{2}} + \Delta v_{i+\frac{3}{2}}) & \text{if} \quad -1 \leq \theta_i^+, \theta_{i+1}^- < 1
\end{cases}
$$

Explicit weights: SP-WENO

Stability estimate:

$$\left| [\![v]\!]_{i+\frac{1}{2}} \right| \leq 2 \left| \Delta v_{i+\frac{1}{2}} \right| \quad \forall \, i \in \mathbb{Z}$$

Explicit weights: SP-WENO

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For ENO-2, the bounding constant $= 2$
For ENO-3, the bounding constant $= 3.5$

Explicit weights: SP-WENO

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For ENO-2, the bounding constant $= 2$
For ENO-3, the bounding constant $= 3.5$

Better stability bounds for equivalent accuracy!

Testing reconstruction accuracy

Consider

$$u(x) = \sin(10\pi x) + x, \quad x \in [0,1].$$

Error in the interface values

$$\|u^-_{i+\frac{1}{2}} - u(x_{i+\frac{1}{2}})\|_{L^p_h} + \|u^+_{i+\frac{1}{2}} - u(x_{i+\frac{1}{2}})\|_{L^p_h}, \quad p \in [1,\infty]$$

where

$$\|(.)_i\|_{L^p_h} = \left(\sum_{i=1}^{N} |(.)_i|^p h\right)^{\frac{1}{p}} \quad \text{for } p < \infty, \qquad \|(.)_i\|_{L^\infty_h} = \max_i |(.)_i|$$

Testing reconstruction accuracy

N	SP-WENO L_h^1 error	SP-WENO L_h^1 rate	ENO3 L_h^1 error	ENO3 L_h^1 rate
40	8.59e-02	-	3.95e-02	-
80	6.73e-03	3.67	4.90e-03	3.01
160	5.01e-04	3.75	6.08e-04	3.01
320	3.64e-05	3.78	7.57e-05	3.01
640	2.59e-06	3.81	9.47e-06	3.00
1280	1.82e-07	3.83	1.18e-06	3.01
2560	1.26e-08	3.85	1.47e-07	3.00

N	WENO3 L_h^1 error	WENO3 L_h^1 rate	ENO2 L_h^1 error	ENO2 L_h^1 rate
40	2.04e-01	-	2.35e-01	-
80	4.03e-02	2.34	5.39e-02	2.12
160	7.25e-03	2.48	1.29e-02	2.07
320	1.18e-03	2.62	3.14e-03	2.03
640	1.77e-04	2.74	7.76e-04	2.02
1280	2.13e-05	3.05	1.93e-04	2.01
2560	2.10e-06	3.34	4.81e-05	2.00

Evolution problems

TeCNO-4 flux used:

- **EC:** Fourth order entropy conservative flux

$$F^{*,4} = \frac{4}{3}F^*(u_i, u_{i+1}) - \frac{1}{6}\big(F^*(u_{i-1}, u_{i+1}) + F^*(u_i, u_{i+2})\big)$$

- **ES:** SP-WENO, ENO-2 or ENO-3 (all preserve sign)

We choose

$$\eta(u) = \frac{u^2}{2} \quad \implies \quad v(u) = u$$

Time integration using a Strong Stability Preserving RK3 scheme

Linear advection

$$u_t + u_x = 0$$

Test 1:

$$\Omega = [-\pi, \pi], \quad T_f = 0.5, \quad CFL = 0.4, \quad u_0(x) = \sin(x)$$

	SP-WENO		ENO3		ENO2	
N	L_h^1		L_h^1		L_h^1	
	error	rate	error	rate	error	rate
50	6.22e-04	-	2.58e-04	-	1.61e-02	-
100	6.90e-05	3.17	3.23e-05	3.00	4.36e-03	1.88
200	7.66e-06	3.17	4.04e-06	3.00	1.16e-03	1.91
400	8.29e-07	3.21	5.05e-07	3.00	3.08e-04	1.91
600	2.26e-07	3.20	1.50e-07	3.00	1.41e-04	1.92
800	8.72e-08	3.31	6.31e-08	3.00	8.09e-05	1.93

Linear advection

$$u_t + u_x = 0$$

Test 2:

$$\Omega = [-\pi, \pi], \quad T_f = 0.5, \quad CFL = 0.5, \quad u_0(x) = \sin^4(x)$$

MUSCL scheme using ENO known to perform poorly.

N	SP-WENO		ENO3		ENO2	
	L^1_h		L^1_h		L^1_h	
	error	rate	error	rate	error	rate
100	1.32e-03	-	1.48e-03	-	2.13e-02	-
200	1.48e-04	3.16	1.97e-04	2.91	6.12e-03	1.80
400	1.64e-05	3.17	2.57e-05	2.94	1.66e-03	1.89
600	4.61e-06	3.14	8.35e-06	2.77	7.63e-04	1.91
800	1.79e-06	3.29	4.86e-06	1.88	4.41e-04	1.90
1000	8.55e-07	3.31	3.62e-06	1.32	2.87e-04	1.92

Similar behaviour observed with TeCNO4.

$$u_t + uu_x = 0, \qquad a_{i+\frac{1}{2}} = \frac{|u_i| + |u_{i+1}|}{2}$$

Test 1:

$$\Omega = [-1, 1], \quad T_f = 0.3, \quad CFL = 0.4, \quad u_0(x) = 1 + \frac{1}{2}\sin(\pi x)$$

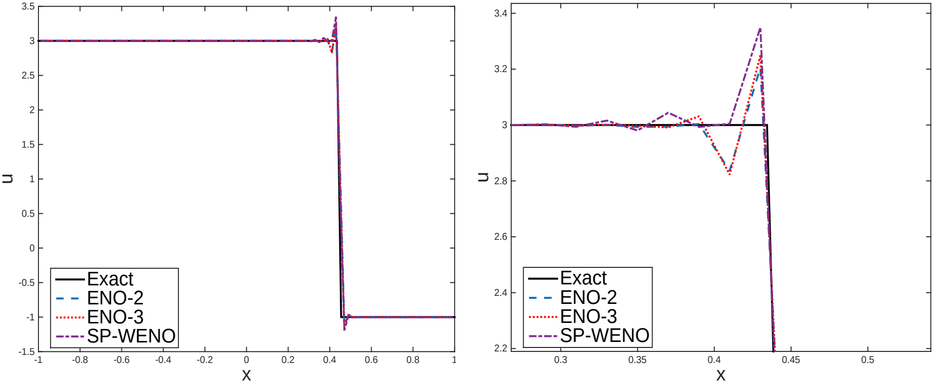
N	SP-WENO		ENO3		ENO2	
	L_h^1		L_h^1		L_h^1	
	error	rate	error	rate	error	rate
50	3.41e-04	-	3.07e-04	-	4.73e-03	-
100	4.17e-05	3.03	4.76e-05	2.69	1.35e-03	1.81
200	4.51e-06	3.21	8.44e-06	2.49	3.77e-04	1.84
400	4.98e-07	3.18	1.80e-06	2.23	1.02e-04	1.89
600	1.33e-07	3.26	7.29e-07	2.23	4.71e-05	1.90
800	5.22e-08	3.25	3.91e-07	2.17	2.72e-05	1.92

Burgers' equations

Test 2: Left moving shock

$$\Omega = [-1, 1], \quad T_f = 0.45, \quad CFL = 0.4$$

$$u_0(x) = \begin{cases} 3 & \text{if } x < 0 \\ -1 & \text{if } x > 0. \end{cases}$$

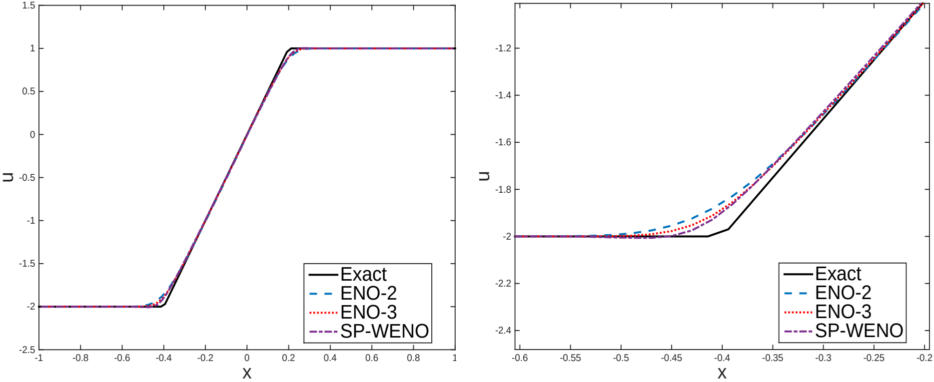


Burgers' equations

Test 3: Rarefaction

$$\Omega = [-1, 1], \quad T_f = 0.2, \quad CFL = 0.4$$

$$u_0(x) = \begin{cases} -2 & \text{if } x < 0 \\ 1 & \text{if } x > 0. \end{cases}$$



Conclusion

Proposed a WENO method satisfying

- Sign property (needed for entropy stability)
- Symmetries (mirror, negation)
- Third (or higher) order accurate
- Good control on reconstruction jumps at interfaces

Conclusion

Proposed a WENO method satisfying

- Sign property (needed for entropy stability)
- Symmetries (mirror, negation)
- Third (or higher) order accurate
- Good control on reconstruction jumps at interfaces

What's next?

- Higher order WENO with similar properties?
- Smooth weights?

Thank You

Constructing an entropy stable scheme

Choosing

$$\eta(u) = \frac{u^2}{2} \quad \Longrightarrow \quad v(u) = u$$

Examples:

- Linear advection: $f(u) = cu$

$$F^*_{i+\frac{1}{2}} = c\frac{u_i + u_{i+1}}{2}$$

- Burgers' equation: $f(u) = u^2/2$

$$F^*_{i+\frac{1}{2}} = \frac{u_i^2 + u_{i+1}^2 + u_i u_{i+1}}{6}$$

Entropy stable scheme for 1D system

Choose flux

$$\mathbf{F}_{i+\frac{1}{2}} = \mathbf{F}^{*,2p}_{i+\frac{1}{2}} - \frac{1}{2}\mathbf{D}_{i+\frac{1}{2}}\Delta\mathbf{V}_{i+\frac{1}{2}}, \qquad \mathbf{D} = \mathbf{D}^\top \geq 0$$

Specific choice

$$\mathbf{D}_{i+\frac{1}{2}} = \mathbf{R}_{i+\frac{1}{2}}\Lambda_{i+\frac{1}{2}}\mathbf{R}^\top_{i+\frac{1}{2}}$$

Define

$$\mathbf{Z} = \mathbf{R}^\top_{i+\frac{1}{2}}\mathbf{V} \quad \rightarrow \quad \text{scaled entropy variable}$$

Reconstruct in $\mathbf{Z}$

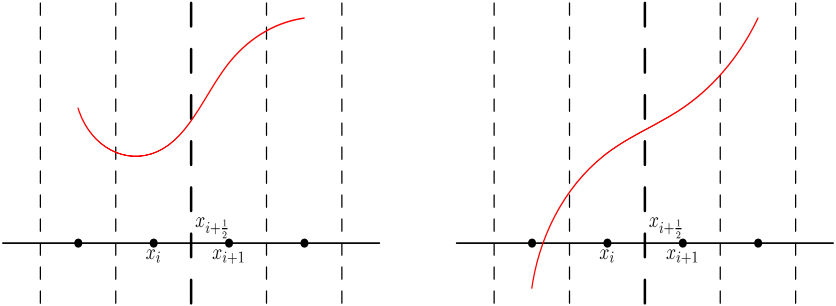
$$\mathbf{F}^{*,2p}_{i+\frac{1}{2}} - \frac{1}{2}\mathbf{R}_{i+\frac{1}{2}}\Lambda_{i+\frac{1}{2}}\Delta\mathbf{Z}_{i+\frac{1}{2}} \quad \longrightarrow \quad \mathbf{F}^{*,2p}_{i+\frac{1}{2}} - \frac{1}{2}\mathbf{R}_{i+\frac{1}{2}}\Lambda_{i+\frac{1}{2}}[\![\mathbf{Z}]\!]_{i+\frac{1}{2}}$$

Sign Property:

$$\text{sign}(\Delta\mathbf{Z}_{i+\frac{1}{2}}) = \text{sign}([\![\mathbf{Z}]\!]_{i+\frac{1}{2}}), \quad \text{componentwise}$$

Essential properties

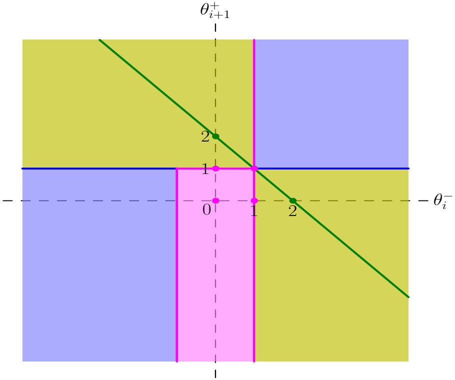
Variable order: If $v''(x) = 0$ for $|x - x_{i+\frac{1}{2}}| = \mathcal{O}(h)$, then ENO-2 polynomials are third order accurate.

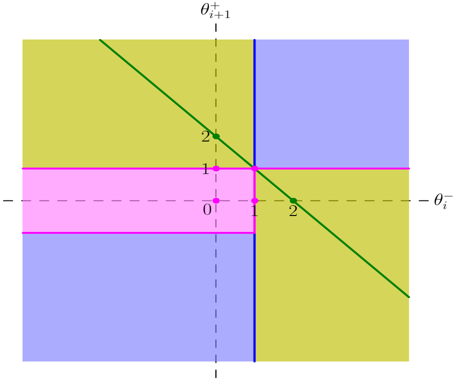


$$C_1, C_2 = \begin{cases} \mathcal{O}(h), & \text{in GC} \\ \text{no restriction}, & \text{in SC} \end{cases}$$

SP-WENO weights

LABEL	Left fig: w_1	Right fig: $\widetilde{w}_0$
blue	0	0
magenta	1	1
green	$\frac{1}{4}$	$\frac{1}{4}$
yellow	$\frac{1}{4}\left(1 - \frac{f^+}{(f^+)^2+(f^-)^2}\right)$	$\frac{1}{4}\left(1 - \frac{f^-}{(f^+)^2+(f^-)^2}\right)$

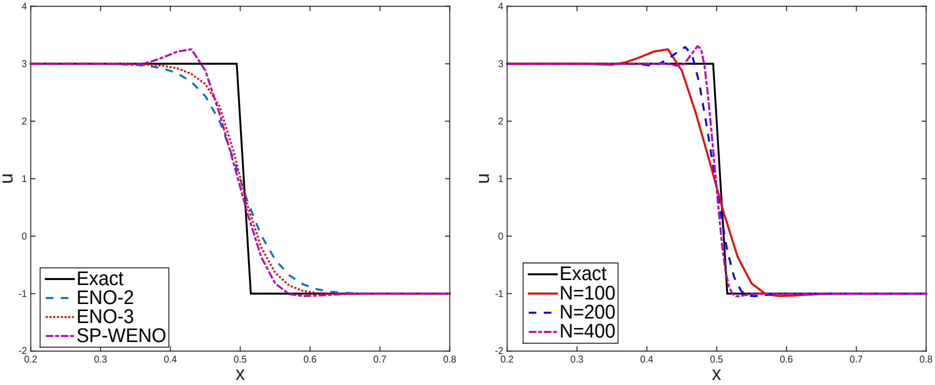




Test 3:

$$\Omega = [-1, 1], \quad T_f = 0.5, \quad CFL = 0.5$$

$$u_0(x) = \begin{cases} 3 & \text{if } x < 0 \\ -1 & \text{if } x > 0. \end{cases}$$



Burgers' equations

$$u_t + uu_x = 0, \qquad a_{i+\frac{1}{2}} = \frac{|u_i| + |u_{i+1}|}{2}$$

Test 1:

- Discontinuity at $t = \frac{2}{\pi} \approx 0.636$.
- The total entropy should be preserved for smooth solution.
- Sharp decrease in entropy after discontuity appears.

$$\frac{E(t) - E(0)}{E(0)}, \qquad E(t) := \sum_i \eta_i(t)h \approx \int_{-1}^{1} \eta(u(x,t))\mathrm{d}x$$

Burgers' equations

$$u_t + uu_x = 0, \qquad a_{i+\frac{1}{2}} = \frac{|u_i| + |u_{i+1}|}{2}$$

Test 1: