Entropy stable schemes for compressible flows on unstructured meshes

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Conservation laws

Consider the (hyperbolic) system

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{f}_1}{\partial x} + \frac{\partial \mathbf{f}_2}{\partial y} = 0 \qquad \forall \quad (\mathbf{x}, t) \in \mathbb{R}^2 \times \mathbb{R}^+$$

$$\mathbf{U}(\mathbf{x}, 0) = \mathbf{U}_0(\mathbf{x}) \qquad \forall \quad \mathbf{x} \in \mathbb{R}^2$$

Examples

- Shallow water equations (Geophysics)
- Euler equations (Aerodynamics)
- MHD equations (Plasma physics)

Non linearities $\implies$ disc. soln. $\implies$ weak (distributional) soln.

Entropy framework

Entropy-entropy flux pair ($\eta(\mathbf{U}),\ \mathbf{q}(\mathbf{U}) \ = \ (q_1(\mathbf{U}), q_2(\mathbf{U}), q_3(\mathbf{U})\ )$

$$\boxed{\partial_t \eta(\mathbf{U}) + \text{div}\cdot(\mathbf{q}(\mathbf{U})) \leq 0}$$

where $\mathbf{V} = \eta'(\mathbf{U}) \rightarrow$ entropy variables.

Single out "a" physically relevant solution

$$\frac{\mathrm{d}}{\mathrm{d}t}\int \eta(\mathbf{U})\mathrm{d}\mathbf{x} \leq 0 \quad \Longrightarrow \quad ||\mathbf{U}(.,t)||_{L^2} \leq C$$

No global existence, uniqueness results for generic multidimensional systems

Entropy stable schemes

- Cartesian meshes
  - Tadmor (1987): Entropy conservative flux.
  - Lefloch et al. (2002): Higher order entropy conservative schemes.
  - Explicit entropy stable schemes by Roe and Ismail (2009), PC (2013).
  - Fjordholm et al. (2011): Entropy stable ENO schemes (TeCNO).
- Madrane et al. (2012): First order entropy stable schemes on unstructured meshes.

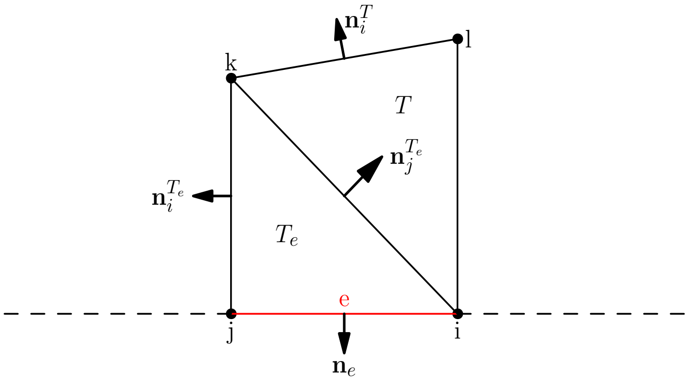
Outline

- 2D conservation laws:
  - Discretisation (unstructured)
  - Entropy conservative/stable schemes
  - High-order schemes (sign property)
  - Numerical results
- 2D Navier-Stokes equations:
  - Global entropy relations
  - Discrete analogue
  - Boundary conditions
  - Numerical results

Discretization

Discretize domain into triangles T, with nodes i, j, k,etc

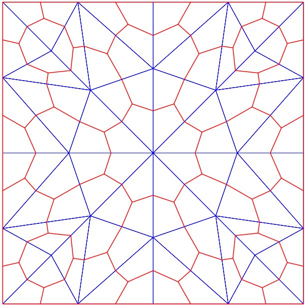
- Primary cell:



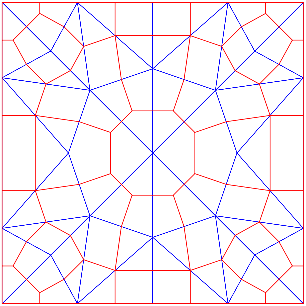
Discretization

Discretize domain into triangles T, with nodes i, j, k,etc

- Primary cell:
- Dual control volumes C_i



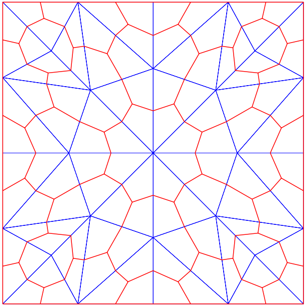
Median dual



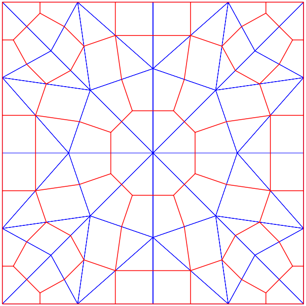
Voronoi dual

Discretize domain into triangles T, with nodes i, j, k,etc

- Primary cell:
- Dual control volumes C_i



Median dual



Voronoi dual

Suitable for complex geometries!!

Semi-discrete scheme

$$\frac{\mathrm{d}\mathbf{U}_i}{\mathrm{d}t} + \frac{1}{|C_i|}\sum_{j\in i}\mathbf{F}_{ij} = 0$$

$\mathbf{U}_i \longrightarrow$ cell average over C_i

$\mathbf{F}_{ij} = \mathbf{F}(\mathbf{U}_i, \mathbf{U}_j, \mathbf{n}_{ij})$ is the numerical flux satisfying

1. Consistency:

$$\mathbf{F}(\mathbf{U}, \mathbf{U}, \mathbf{n}) = \mathbf{F}(\mathbf{U}, \mathbf{n}) := \mathbf{f}_1(\mathbf{U})n_1 + \mathbf{f}_2(\mathbf{U})n_2$$

2. Conservation:

$$\mathbf{F}(\mathbf{U}_1, \mathbf{U}_2, \mathbf{n}) = -\mathbf{F}(\mathbf{U}_2, \mathbf{U}_1, -\mathbf{n}) \qquad \forall \quad \mathbf{U}_1, \mathbf{U}_2, \mathbf{n}$$

First order entropy stable flux [1]

- Entropy conservative flux

$$\mathbf{F}_{ij} = \mathbf{F}^*_{ij} \quad \text{s.t.} \quad \frac{\mathrm{d}\eta(\mathbf{U}_i)}{\mathrm{d}t} + \frac{1}{|C_i|}\sum_{j\in i} q^*_{ij} = 0$$

- Sufficient condition (Tadmor)

$$\Delta\mathbf{V}^\top_{ij}\mathbf{F}^*_{ij} = \psi(\mathbf{U}_j, \mathbf{n}_{ij}) - \psi(\mathbf{U}_i, \mathbf{n}_{ij})$$

where

$$\psi(\mathbf{U},\mathbf{n}) := \mathbf{V}(\mathbf{U})^\top\mathbf{F}(\mathbf{U},\mathbf{n}) - \underbrace{q(\mathbf{U},\mathbf{n})}_{q_1 n_1 + q_2 n_2} \rightarrow \text{(entropy potential)}$$

- Entropy variable based dissipation

$$\mathbf{F}_{ij} = \mathbf{F}^*_{ij} - \frac{1}{2}\mathbf{D}_{ij}\Delta\mathbf{V}_{ij}, \qquad \mathbf{D}_{ij} = \mathbf{D}^\top_{ij} \geq 0$$

$$\implies \quad \frac{\mathrm{d}\eta(\mathbf{U}_i)}{\mathrm{d}t} + \frac{1}{|C_i|}\sum_{j\in i} q_{ij} \leq 0$$

[1] A. Madrane, UF, SM, and E. Tadmor. *Entropy conservative and entropy stable finite volume schemes for multi-dimensional conservation laws on unstructured meshes.* (in review)

First order entropy stable flux

- Kinetic energy and entropy conservative flux (PC) for Euler equations.
- Dissipation operator

$$\mathbf{D}_{ij} = \mathbf{R}_{ij}\Lambda_{ij}\mathbf{R}_{ij}^{\top}$$

$$\mathbf{R}_{ij} \to \text{scaled eigenvectors of } \mathbf{F}_{\mathbf{U}} \text{ (Barth)}$$

$$\Lambda_{ij} = \text{diag}[|\lambda_1|, ..., |\lambda_n|] \to \text{Roe type}$$

- KEPES $\equiv$ Kinetic energy and entropy conservative flux + Roe dissipation

High-order diffusion

$$\Delta \mathbf{V}_{ij} \sim \mathcal{O}(|\Delta \mathbf{x}_{ij}|) \quad \Longrightarrow \quad \text{first order flux}$$

Idea: For each $\mathbf{x}_{ij}$, reconstruct $\mathbf{V}$ in C_i, C_j with polynomials $\mathbf{V}_i(\mathbf{x}), \mathbf{V}_j(\mathbf{x})$ respectively.

$$\mathbf{V}_{ij} = \mathbf{V}_i(\mathbf{x}_{ij}), \qquad \mathbf{V}_{ji} = \mathbf{V}_j(\mathbf{x}_{ij}), \qquad [\![\mathbf{V}]\!]_{ij} = \mathbf{V}_{ji} - \mathbf{V}_{ij}$$

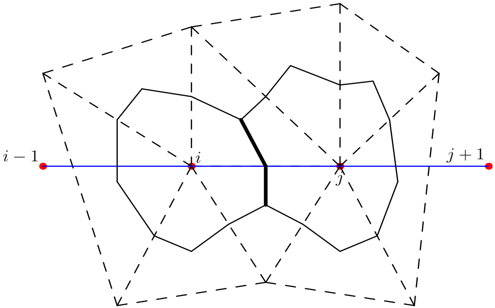
Sign property (Fjordholm et al.)

For each $\mathbf{x}_{ij}$, define the scaled entropy variables $\mathbf{Z} = \mathbf{R}_{ij}^{\top}\mathbf{V}$. Then numerical flux

$$\mathbf{F}_{ij} = \mathbf{F}^*_{ij} - \frac{1}{2}\mathbf{R}_{ij}\Lambda_{ij}[\![\mathbf{Z}]\!]_{ij} = \mathbf{F}^*_{ij} - \frac{1}{2}\mathbf{D}_{ij}[\![\mathbf{V}]\!]_{ij}, \qquad [\![\mathbf{V}]\!]_{ij} = (R_{ij}^T)^{-1}[\![\mathbf{Z}]\!]_{ij}$$

is entropy stable if the sign property holds for $\mathbf{Z}$ componentwise.

$$\text{sign}([\![Z]\!]_{ij}) = \text{sign}(\Delta Z_{ij})$$



Define

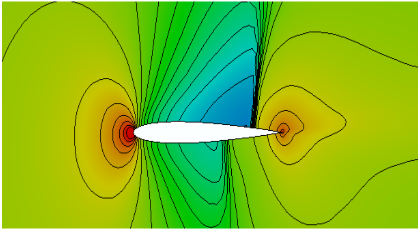
$$\mathbf{Z}_i = \mathbf{R}_{ij}^\top \mathbf{V}_i, \quad \mathbf{Z}_j = \mathbf{R}_{ij}^\top \mathbf{V}_j$$

The componentwise reconstructed scaled variables are

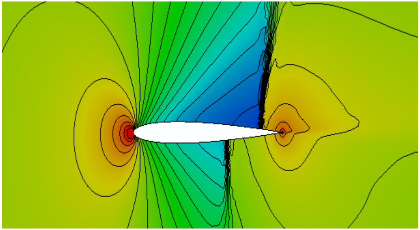
$$\left.\begin{aligned} Z_{ij} &= Z_i + \frac{1}{2}\text{minmod}\left(\Delta_{ij}^f, \Delta_{ij}^b\right) \\ Z_{ji} &= Z_j - \frac{1}{2}\text{minmod}\left(\Delta_{ji}^f, \Delta_{ji}^b\right) \end{aligned}\right\} \quad \text{need } \nabla_h \mathbf{Z}_i = \mathbf{R}_{ij}^\top \nabla_h \mathbf{V}_i$$

Transonic flow past NACA-0012 airfoil

angle of attack $= 2$ degrees, freestream $M = 0.85$

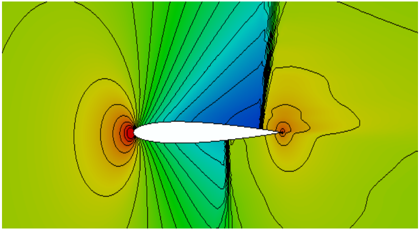


KEPES



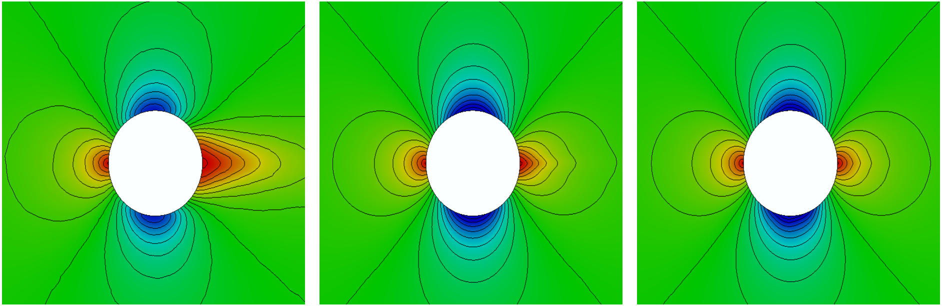
KEPES-TeCNO

Mach number, 20 equally spaced contours between 0.5 and 1.5



ROE (MUSCL)

freestream $M = 0.3$, symmetric, isentropic flow



KEPES KEPES-TeCNO KEPES2

Scheme	Minimum	Maximum	Percent deviation from s_∞	
KEPES	2.07147	2.08695	+0.747 %	-0.000 %
KEPES-TeCNO	2.07147	2.07208	+0.029 %	-0.000 %
KEPES2	2.07139	2.07153	+0.003 %	-0.004 %

Table : Physical entropy bounds, with freestream $s_\infty = 2.07147$

2D Navier-Stokes Equations

Initial boundary valued problem

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{f}_1}{\partial x} + \frac{\partial \mathbf{f}_2}{\partial y} = \frac{\partial \mathbf{g}_1}{\partial x} + \frac{\partial \mathbf{g}_2}{\partial y} \qquad \forall \ \mathbf{x} = (x, y) \in \Omega \subset \mathbb{R}^2, \ t \in \mathbb{R}^+$$

$$\mathbf{U}(\mathbf{x}, 0) = \mathbf{U}_0(\mathbf{x}) \qquad \forall \ \mathbf{x} \in \Omega$$

$$\mathcal{B}(\mathbf{U}(\mathbf{x}, t)) = \mathbf{h}(x, t) \qquad \forall \ \mathbf{x} \in \partial\Omega$$

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Specific entropy pair for symmetrization

$$\eta(\mathbf{U}) = -\frac{\rho s}{\gamma - 1}, \quad q_1(\mathbf{U}) = -\frac{\rho u s}{\gamma - 1}, \quad q_2(\mathbf{U}) = -\frac{\rho v s}{\gamma - 1}$$

2D Navier-Stokes Equations

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Viscous fluxes in terms of entropy variables

$$\mathbf{g}_1 = \mathbf{K}_{11}(\mathbf{V})\frac{\partial \mathbf{V}}{\partial x} + \mathbf{K}_{12}(\mathbf{V})\frac{\partial \mathbf{V}}{\partial y}, \qquad \mathbf{g}_2 = \mathbf{K}_{21}(\mathbf{V})\frac{\partial \mathbf{V}}{\partial x} + \mathbf{K}_{22}(\mathbf{V})\frac{\partial \mathbf{V}}{\partial y}$$

where

$$\mathbf{K} = \begin{bmatrix} \mathbf{K}_{11} & \mathbf{K}_{12} \\ \mathbf{K}_{21} & \mathbf{K}_{22} \end{bmatrix} \geq 0$$

Global entropy relation

Integrating NSE against $\mathbf{V}^\top$ gives us

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \eta &= -\int_{\partial\Omega} q(\mathbf{U},\mathbf{n}) - \int_\Omega \left\langle \mathbf{K}\cdot \widetilde{\nabla}\mathbf{V}, \widetilde{\nabla}\mathbf{V} \right\rangle + \int_{\partial\Omega} \mathbf{V}^\top \mathbf{G}(\mathbf{U},\mathbf{n}) \\ &\leq -\int_{\partial\Omega} q(\mathbf{U},\mathbf{n}) + \int_{\partial\Omega} \mathbf{V}^\top \mathbf{G}(\mathbf{U},\mathbf{n}) \end{aligned}$$

where

$$\mathbf{G}(\mathbf{U},\mathbf{n}) = \mathbf{g}_1(\mathbf{U})n_1 + \mathbf{g}_2(\mathbf{U})n_2$$

and

$$\widetilde{\nabla}\mathbf{V} = \begin{pmatrix} \partial_x \mathbf{V} \\ \partial_y \mathbf{V} \end{pmatrix} \in \mathbb{R}^8 \quad \text{obtained from} \quad \nabla\mathbf{V} = \left(\partial_x \mathbf{V}, \partial_y \mathbf{V}\right) \in \mathbb{R}^{4\times 2}$$

Semi-discrete scheme

Notations

$$\begin{aligned} j \in i &= \{ \text{ all vertices } j \text{ neighbouring vertex } i \ \} \\ i \in T &= \{ \text{ all vertices } i \text{ belonging to triangle } T \ \} \\ T \in i &= \{ \text{ all triangles } T \text{ having vertex } i \ \} \\ \Gamma &= \{ \text{ all boundary edges of the primary mesh } \} \\ \Gamma_i &= \{ \text{ all boundary edges of the primary mesh having vertex } i \ \} \end{aligned}$$

$$|C_i| \frac{\mathrm{d}\mathbf{U}_i}{\mathrm{d}t} = -\sum_{j \in i} \mathbf{F}_{ij} + \sum_{T \in i} \mathbf{G}^T \cdot \frac{\mathbf{n}_i^T}{2} - \sum_{e \in \Gamma_i} \mathbf{F}_{ie} + \sum_{e \in \Gamma_i} \mathbf{G}^e \cdot \frac{\mathbf{n}_e}{2}$$

Choosing flux terms

- Interior inviscid flux $\mathbf{F}_{ij}$ is an entropy stable flux

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- Interior viscous flux $\mathbf{G}^T = (\mathbf{G}_1^T, \mathbf{G}_2^T)$ is chosen as

$$\mathbf{G}_\alpha^T = \mathbf{K}_{\alpha 1}^T \cdot \partial_x^h \mathbf{V}^T + \mathbf{K}_{\alpha 2}^T \cdot \partial_y^h \mathbf{V}^T, \qquad \alpha = 1, 2$$

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- Boundary inviscid flux $\mathbf{F}_{ie}$ is chosen as

$$\mathbf{F}_{ie} = \mathbf{F}_{ie}\left(\mathbf{U}, \mathbf{U}^b, \mathbf{n}_e\right)$$

$$\mathbf{F}_{ie} = \begin{bmatrix} F_{ie}^\rho \\ \frac{1}{2}(p_i^b \mathbf{n}_e) + \mathbf{u}_i F_{ie}^\rho \\ F_{ie}^E \end{bmatrix}, \qquad F_{ie}^\rho = \left(\rho \frac{u_{\mathbf{n}_e}^b}{2}\right)_i$$

$$F_{ie}^E = \left[\left(\rho \frac{|\mathbf{u}|^2}{2} + \frac{\gamma p}{\gamma - 1}\right) \frac{u_{\mathbf{n}_e}^b}{2}\right]_i + \left[(p^b - p) \frac{u_{\mathbf{n}_e}}{2}\right]_i$$

BC implemented weakly

Choosing flux terms

- Interior inviscid flux $\mathbf{F}_{ij}$ is an entropy stable flux
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- Boundary inviscid flux $\mathbf{F}_{ie}$ is chosen as

$$\mathbf{F}_{ie} = \mathbf{F}_{ie}\left(\mathbf{U}, \mathbf{U}^b, \mathbf{n}_e\right)$$

- Boundary viscous flux $\mathbf{G}^e = (\mathbf{G}_1^e, \mathbf{G}_2^e)$ is chosen as

$$\mathbf{G}^e \cdot \mathbf{n}_e = \mathbf{G}^{T_e} \cdot \mathbf{n}_e + \mathbf{C}^e$$

Discrete global entropy relation

Pre-multiplying the scheme by $\mathbf{V}_i$ and summing over all nodes.

$$\frac{\mathsf{d}}{\mathsf{d}t}\sum_i \eta_i |C_i| \leq -\sum_{e\in\Gamma}\left[-\left(\frac{\rho s}{\gamma-1}\frac{u^b_{\mathbf{n}_e}}{2}\right)_i - \left(\frac{\rho s}{\gamma-1}\frac{u^b_{\mathbf{n}_e}}{2}\right)_j\right] + \sum_{e\in\Gamma}\left[\frac{(\mathbf{V}^b_i+\mathbf{V}^b_j)^\top}{2}\mathbf{G}^{T_e}\cdot\mathbf{n}_e + \frac{(\mathbf{V}_i+\mathbf{V}_j)^\top}{2}\mathbf{C}^e\right]$$

which is consistent with

$$\frac{\mathsf{d}}{\mathsf{d}t}\int_\Omega \eta \leq -\int_{\partial\Omega} q(\mathbf{U},\mathbf{n}) + \int_{\partial\Omega}\mathbf{V}^\top\mathbf{G}(\mathbf{U},\mathbf{n})$$

Homogeneous boundary conditions

Slip BC (Euler)

Continuous setup	Discrete setup
$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega} \eta \leq 0$$	$$\frac{\mathrm{d}}{\mathrm{d}t}\sum_{i} \eta_i \vert C_i \vert \leq 0$$

Homogeneous boundary conditions

Slip BC (Euler)

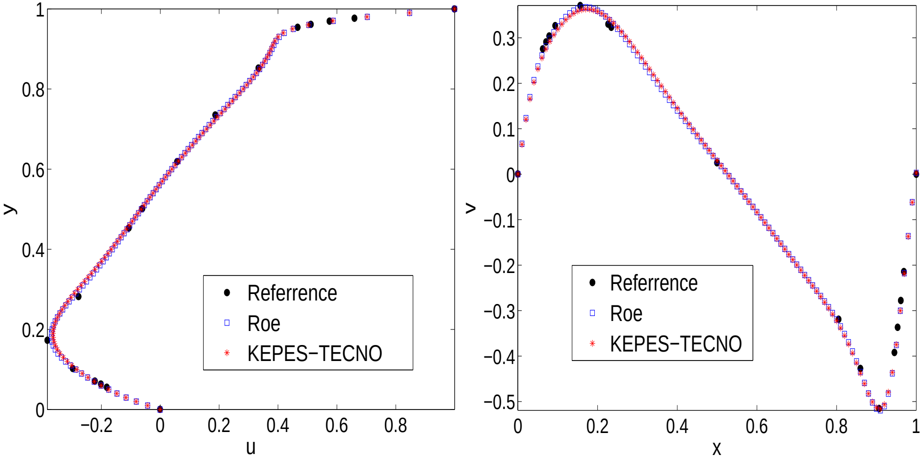
Continuous setup	Discrete setup
$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \eta \leq 0$	$\frac{\mathrm{d}}{\mathrm{d}t}\sum_i \eta_i \lvert C_i \rvert \leq 0$

No-slip BC with zero heat flux

Continuous setup	Discrete setup
$\mathbf{G}^e \cdot \mathbf{n}_e = \begin{pmatrix} 0 \\ \boldsymbol{\tau} \cdot \mathbf{n}_e \\ 0 \end{pmatrix}$	$\mathbf{C}^e = -\begin{pmatrix} 0 & 0 & 0 & \left(\frac{\kappa}{R(V^{e,(4)})^2}\left(\nabla_h \mathbf{V}^{T_e,(4)} \cdot \mathbf{n}_e\right)\right)\end{pmatrix}^\top$
$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \eta \leq 0$	$\mathbf{G}^e \cdot \mathbf{n}_e = \begin{pmatrix} 0 \\ \boldsymbol{\tau}^h \cdot \mathbf{n}_e \\ 0 \end{pmatrix}, \quad \frac{\mathrm{d}}{\mathrm{d}t}\sum_i \eta_i \lvert C_i \rvert \leq 0$

Numerical results: Lid driven cavity

Top lid moving with velocity $(u = 1, v = 0)$. No-slip BC on other 3 faces.
Comparing with data of Ghia et al.



u on vert. line through center

v on hor. line through center

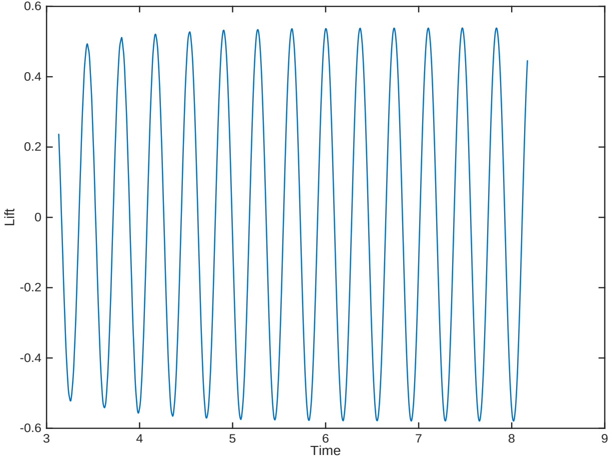
Flow past a cylinder

Inflow: $$\left(u = 4u_m \frac{y(H-y)}{H^2}, 0\right), \quad u_m = 1.5, \qquad Re = 100$$

back

Flow past a cylinder

Inflow: $\left(u = 4u_m \frac{y(H-y)}{H^2}, 0\right), \quad u_m = 1.5, \qquad Re = 100$



Conclusion

- Second order (limited) entropy stable finite volume scheme for system of conservation laws, on unstructured meshes.
- Discretized viscous fluxes for Navier-Stokes equations with appropriate boundary (homogeneous) fluxes to obtain global entropy stability.

Next steps:

- Higher order finite volume schemes?
- Entropy stable boundary conditions for general BC?
- Parellelisation and 3D implementation.

Thank you.

2D Euler Equations

$$\mathbf{U} = \begin{bmatrix} \rho \\ \rho u \\ \rho v \\ E \end{bmatrix}, \qquad \mathbf{f}_1(\mathbf{U}) = \begin{bmatrix} \rho u \\ \rho u^2 + p \\ \rho u v \\ u(E+p) \end{bmatrix}, \quad \mathbf{f}_2(\mathbf{U}) = \begin{bmatrix} \rho v \\ \rho u v \\ \rho v^2 + p \\ v(E+p) \end{bmatrix}$$

where

$$E = \rho\left(\frac{1}{2}u^2 + \frac{p}{(\gamma-1)\rho}\right)$$

Choose

$$\eta(\mathbf{U}) = -\frac{\rho s}{\gamma - 1}, \quad q_1(\mathbf{U}) = -\frac{\rho u s}{\gamma - 1}, \quad q_2(\mathbf{U}) = -\frac{\rho v s}{\gamma - 1}$$

$$s = \ln(p) - \gamma \ln(\rho) \quad \text{(physical entropy)}$$

$$\mathbf{V} = \begin{bmatrix} \frac{\gamma - s}{\gamma - 1} - \beta|\mathbf{u}|^2 \\ 2\beta\mathbf{u} \\ -2\beta \end{bmatrix}, \qquad \beta = \frac{\rho}{2p} = \frac{1}{RT}$$

Entropy conservative flux for Euler

Kinetic energy and entropy conservative flux

$$\mathbf{F}^* = \begin{bmatrix} F^{*,\rho} \\ F^{*,m_1} \\ F^{*,m_2} \\ F^{*,e} \end{bmatrix} = \begin{bmatrix} \widehat{\rho}\overline{u}_n \\ \tilde{p}n_1 + \overline{u}F^{*,\rho} \\ \tilde{p}n_2 + \overline{v}F^{*,\rho} \\ F^{*,e} \end{bmatrix}$$

where

$$\overline{u}_n = \overline{u}n_1 + \overline{v}n_2, \qquad \tilde{p} = \frac{\overline{\rho}}{2\overline{\beta}},$$

$$F^{*,e} = \left[\frac{1}{2(\gamma - 1)\widehat{\beta}} - \frac{1}{2}\overline{|\mathbf{u}|^2} \right] F^{*,\rho} + \overline{\mathbf{u}} \cdot \mathbf{F}^{*,m}$$

The crucial property for kinetic energy preservation (Jameson)

$$\mathbf{F}^m = p\mathbf{n} + \overline{\mathbf{u}}F^\rho$$

for any consistent approximations of p and F^ρ.

Entropy stable KEPES flux

For Euler equations, we choose **Roe type dissipation**

$$\mathbf{D}_{ij} = \mathbf{R}_{ij} |\Lambda_{ij}| \mathbf{R}_{ij}^\top$$

where

$$\mathbf{R} = \begin{bmatrix} 1 & 1 & 0 & 1 \\ u - a\tilde{n}_1 & u & \tilde{n}_2 & u + a\tilde{n}_1 \\ v - a\tilde{n}_2 & v & -\tilde{n}_1 & v + a\tilde{n}_2 \\ H - a u_{\tilde{n}} & \frac{1}{2}|\mathbf{u}|^2 & u\tilde{n}_2 - v\tilde{n}_1 & H + a u_{\tilde{n}} \end{bmatrix} \mathbf{S}^{\frac{1}{2}}$$

$$\mathbf{S} = \text{diag}\left[\frac{\rho}{2\gamma}, \quad \frac{(\gamma-1)\rho}{\gamma}, \quad p, \quad \frac{\rho}{2\gamma}\right]$$

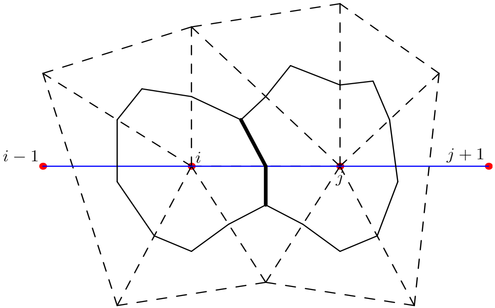
$$\Lambda = \text{diag}\left[u_n - a, \quad u_n, \quad u_n, \quad u_n + a\right]$$

$\tilde{\mathbf{n}} \rightarrow$ unit face normal, $\quad u_{\tilde{n}} = \mathbf{u} \cdot \tilde{\mathbf{n}}$

$a = \sqrt{\frac{\gamma p}{\rho}} \rightarrow$ speed of sound in air

$H = \frac{a^2}{\gamma - 1} + \frac{|\mathbf{u}|^2}{2} \rightarrow$ specific enthalpy

Second order (limited) reconstruction



- The forward differences

$$\Delta^f_{ij} = Z_j - Z_i, \qquad \Delta^f_{ji} = Z_{j+1} - Z_j = [Z_i + 2(\mathbf{x}_j - \mathbf{x}_i)^\top \nabla_h Z_j] - Z_j$$

- The backward differences

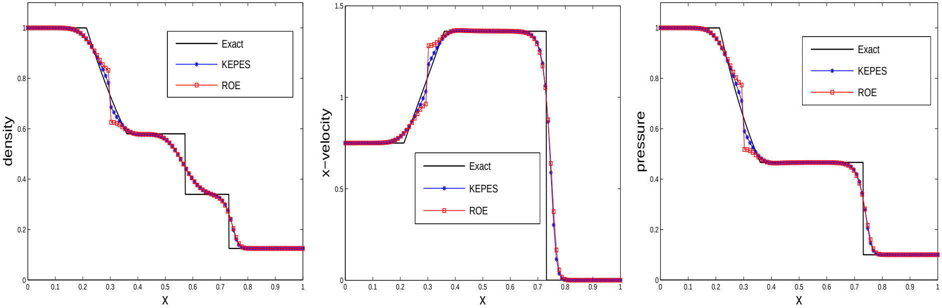
$$\Delta^b_{ij} = Z_i - Z_{i-1} = Z_i - [Z_j - 2(\mathbf{x}_j - \mathbf{x}_i)^\top \nabla_h Z_i], \qquad \Delta^b_{ji} = Z_j - Z_i$$

$$\nabla_h \mathbf{Z}_i = \mathbf{R}^\top_{ij} \nabla_h \mathbf{V}_i$$

Entropy stability is important

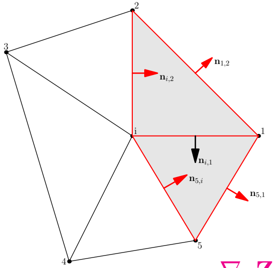
1D modified shocktube test:

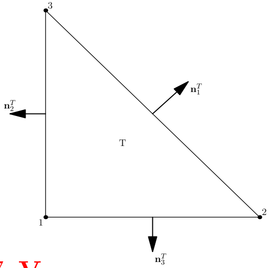
$$(\rho_L, u_L, p_L) = (1.0, 0.75, 1.0), \quad (\rho_R, u_R, p_R) = (0.125, 0.0, 0.1)$$



Roe's solver (without fix) gives entropy violating shock.

Gradients



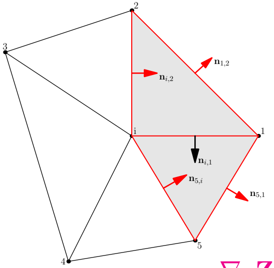


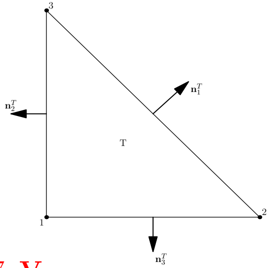
$$\nabla_h \mathbf{Z}_i = \mathbf{R}_{ij}^\top \nabla_h \mathbf{V}_i$$

$$\nabla_h \mathbf{V}^T = \frac{1}{|T|}\left[\underbrace{\frac{(\mathbf{V}_i + \mathbf{V}_j)}{2} \otimes \mathbf{n}_k^T}_{\text{trapezoidal rule}} + \frac{(\mathbf{V}_j + \mathbf{V}_k)}{2} \otimes \mathbf{n}_i^T + \frac{(\mathbf{V}_k + \mathbf{V}_i)}{2} \otimes \mathbf{n}_j^T\right]$$

$$\nabla_h \mathbf{V}^{T_e} = \frac{1}{|T|}\left[\frac{(\mathbf{V}_i^b + \mathbf{V}_j^b)}{2} \otimes \mathbf{n}_e + \frac{(\mathbf{V}_j + \mathbf{V}_k)}{2} \otimes \mathbf{n}_i^{T_e} + \frac{(\mathbf{V}_k + \mathbf{V}_i)}{2} \otimes \mathbf{n}_j^{T_e}\right]$$

Gradients





$$\nabla_h \mathbf{Z}_i = \mathbf{R}_{ij}^\top \nabla_h \mathbf{V}_i$$

$$\nabla_h \mathbf{V}_i = \frac{\sum_{T \in i} |T| \nabla_h \mathbf{V}^T}{\sum_{T \in i} |T|}$$

Time integration:

- SSP-RK3: Explicit strong stability preserving RK3

$$\frac{\mathrm{d}u}{\mathrm{d}t} = L(u)$$

$$\begin{aligned}
v^{(0)} &= v^n \\
v^{(1)} &= v^{(0)} + \Delta t L(v^{(0)} \\
v^{(2)} &= \frac{3}{4}v^{(0)} + \frac{1}{4}\left[v^{(1)} + \Delta t L(v^{(1)}\right] \\
v^{(3)} &= \frac{1}{3}v^{(0)} + \frac{2}{3}\left[v^{(2)} + \Delta t L(v^{(2)}\right] \\
v^{n+1} &= v^{(3)}
\end{aligned}$$

Step in wind tunnel

- Supersonic flow past forward step, M=3.
- Corner is the center of a rarefaction fan $\implies$ singular point.
- Shocks undergo reflections.

2D Navier-Stokes Equations

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{f}_1}{\partial x} + \frac{\partial \mathbf{f}_2}{\partial y} = \frac{\partial \mathbf{g}_1}{\partial x} + \frac{\partial \mathbf{g}_2}{\partial y}$$

$$\mathbf{g}_1(\mathbf{U}) = \begin{bmatrix} 0 \\ \tau_{11} \\ \tau_{21} \\ u\tau_{11} + v\tau_{21} - Q_1 \end{bmatrix}, \qquad \mathbf{g}_2(\mathbf{U}) = \begin{bmatrix} 0 \\ \tau_{21} \\ \tau_{22} \\ u\tau_{21} + v\tau_{22} - Q_2 \end{bmatrix}$$

where

$$\tau_{11} = \frac{4\mu}{3}\frac{\partial u}{\partial x} - \frac{2\mu}{3}\frac{\partial v}{\partial y}, \qquad \tau_{12} = \tau_{21} = \mu\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right),$$

$$\tau_{22} = \frac{4\mu}{3}\frac{\partial v}{\partial y} - \frac{2\mu}{3}\frac{\partial u}{\partial x}, \qquad \mathbf{Q} = (Q_1, Q_2) = -\kappa \nabla T$$

$\mu \longrightarrow$ dynamic coeff. of viscosity

$\kappa \longrightarrow$ coeff. of heat conductance

Inviscid boundary flux

Boundary inviscid flux $\mathbf{F}_{ie}$ is chosen as

$$\mathbf{F}_{ie} = \begin{bmatrix} F^{\rho}_{ie} \\ \frac{1}{2}(p^b_i \mathbf{n}_e) + \mathbf{u}_i F^{\rho}_{ie} \\ F^{E}_{ie} \end{bmatrix}, \qquad F^{\rho}_{ie} = \left(\rho \frac{u^b_{\mathbf{n}_e}}{2} \right)_i$$

$$F^{E}_{ie} = \left[\left(\rho \frac{|\mathbf{u}|^2}{2} + \frac{\gamma p}{\gamma - 1} \right) \frac{u^b_{\mathbf{n}_e}}{2} \right]_i + \left[(p^b - p) \frac{u_{\mathbf{n}_e}}{2} \right]_i$$

Numerical results: Viscous shock tube problem

Balancing effects of physical viscosity, the heat flux and artificial numerical viscosity

$$(\rho_L, u_L, p_L) = (1.0, 0.0, 1.0) \qquad (\rho_R, u_R, p_R) = (0.125, 0.0, 0.1)$$

$$Pr = 1.0, \qquad h \sim 1.0 \times 10^{-2}$$

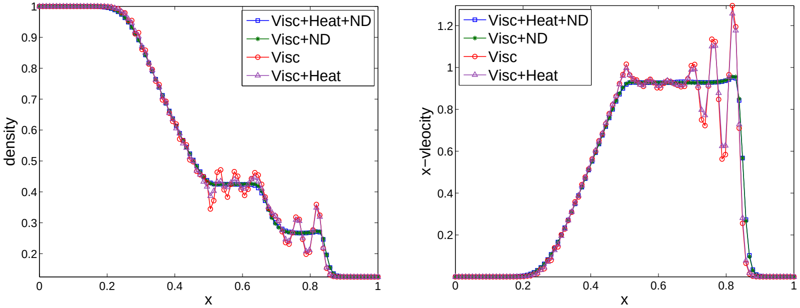
$$\kappa = \frac{\mu c_p}{Pr}$$

We consider three flows regimes depending on μ.

ND	$\rightarrow$	Numerical dissipation present
Visc	$\rightarrow$	Physical viscosity present
Heat	$\rightarrow$	Heat flux present

Numerical results: Viscous shock tube problem

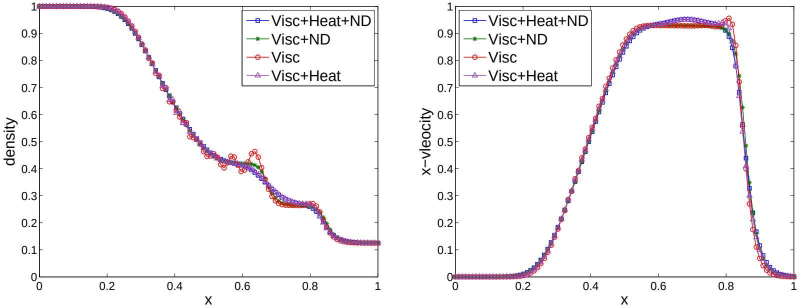
Regime 1: $\mu = 2.0 \times 10^{-4}$



Severe oscillations in the absence of ND.

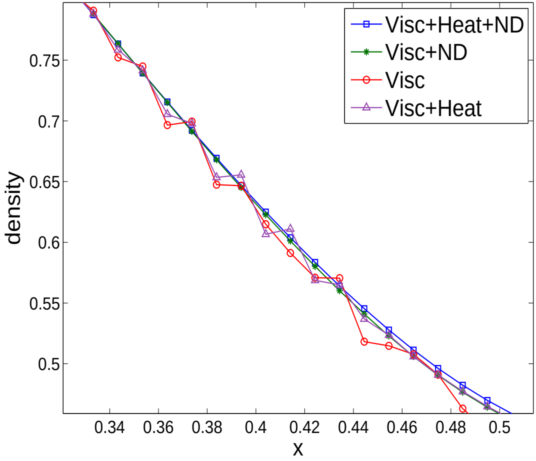
Numerical results: Viscous shock tube problem

Regime 2: $\mu = 2.0 \times 10^{-3}$



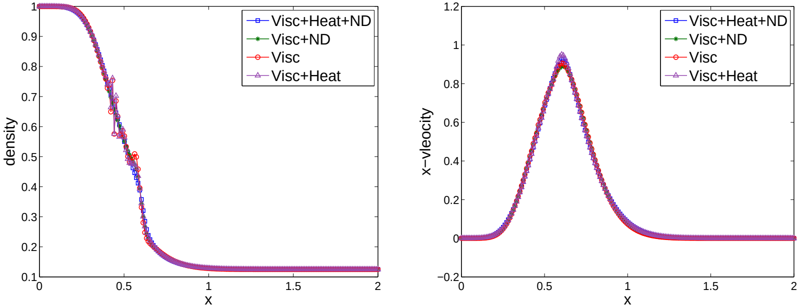
Numerical results: Viscous shock tube problem

Regime 2: $\mu = 2.0 \times 10^{-3}$



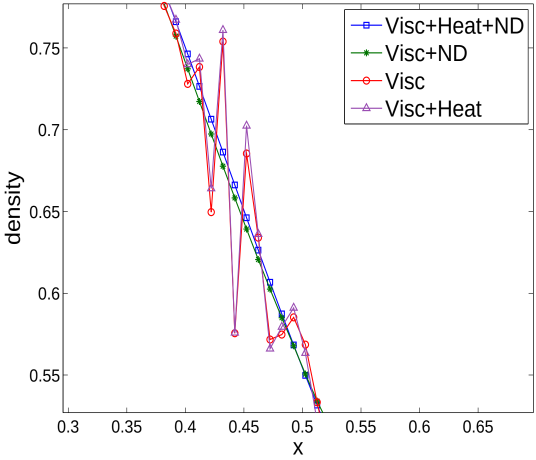
Numerical results: Viscous shock tube problem

Regime 3: $\mu = 2.0 \times 10^{-2}$



Numerical results: Viscous shock tube problem

Regime 2: $\mu = 2.0 \times 10^{-2}$



Numerical results: Laminar flat-plate boundary layer

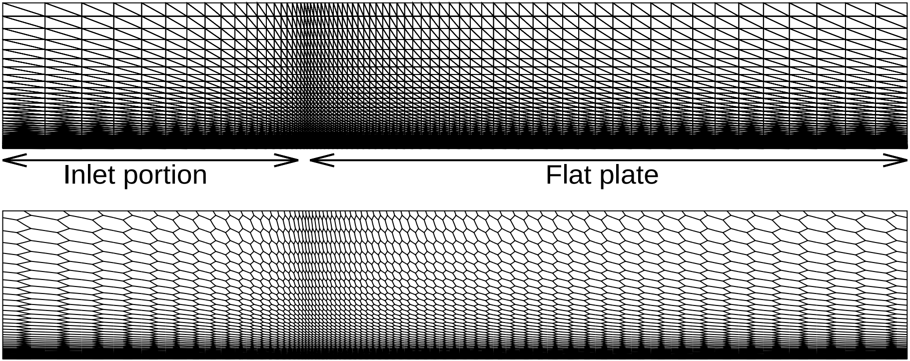
Re $= 10^5$, $M_\infty = 0.1$

Slip BC on inlet portion

Farfield BC on In-flow

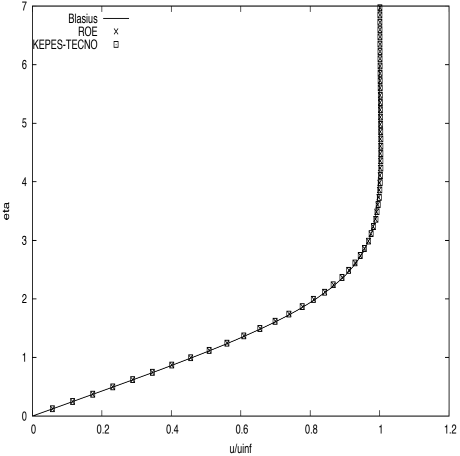
Adiabatic no-slip BC on flat plate

Freestream pressure BC on top and outlet

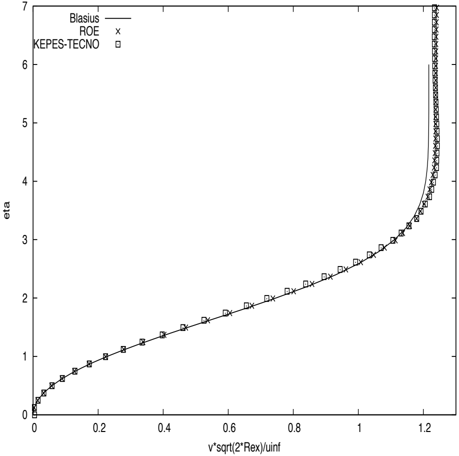


Numerical results: Laminar flat-plate boundary layer

Comparing with Blasius semi-analytical solution on the vertical line through plate center



Streamwise velocity



Vertical velocity